

Quantitative Mode Stability for the Wave Equation on the Kerr Spacetime

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Abstract

We give a quantitative refinement and simple proofs of mode stability type statements for the wave equation on Kerr backgrounds in the full sub-extremal range ($|a| < M$). As an application, we are able to quantitatively control the energy flux along the horizon and null infinity and establish integrated local energy decay for solutions to the wave equation in any bounded-frequency regime.

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1 Introduction

One of the most central problems in mathematical General Relativity is the non-linear stability of the 2-parameter family of Kerr spacetimes $(\mathcal{M}, g_{a,M})$, indexed by mass M and specific angular momentum a . Though the full non-linear problem (the stability of $(\mathcal{M}, g_{a,M})$ as a family of solutions to the Einstein vacuum equations $\text{Ric}(g) = 0$) appears intractable at the moment, much work has been done in the linear setting. In particular, experience teaches us that resolving the non-linear problem will require a robust understanding of decay for solutions of the wave equation $\square_g \psi = 0$ on the fixed Kerr spacetime (\mathcal{M}, g) . Let us direct the reader to the lecture notes [9] for a general introduction to linear waves on black hole backgrounds.

Surprisingly, even the most basic boundedness and decay statements for the wave equation on Kerr remained unanswered until quite recently. Boundedness and decay results for solutions to the wave equation on the 1-parameter Schwarzschild subfamily ($a = 0$) were obtained in [19],[11], and [5]. The first global result for general solutions to the Cauchy problem on a rotating black hole ($a \neq 0$) was obtained in [7] where Dafermos and Rodnianski established uniform boundedness in the case $|a| \ll M$. Following this, decay results, again in the case $|a| \ll M$, were obtained by various authors, e.g. [9], [8], [2], [21], [27], [28], and [23]. For the full sub-extremal range of Kerr black holes ($|a| < M$), the coupling between “superradiance” and trapping presented serious conceptual difficulties; nevertheless, in [10] Dafermos and Rodnianski succeeded in establishing boundedness and decay for the wave equation on a general sub-extremal Kerr background. Their proof re-

quired an additional estimate¹ for the “bounded superradiant frequencies.” This paper provides the needed result.

Interestingly, the problem of the superradiant frequencies will lead us back to the classical mode analysis of the physics literature, see [20] and [31], albeit from a quite different perspective. Mode solutions to the wave equation will be reviewed in section 1.2; for now, we simply recall that a solution ψ to the wave equation $\square_g \psi = 0$ is called a mode solution if

$$\psi(t, r, \theta, \phi) = e^{-i\omega t} e^{im\phi} S(\theta) R(r) \text{ with } \omega \in \mathbb{C} \text{ and } m \in \mathbb{Z},$$

where (t, r, θ, ϕ) are Boyer-Lindquist coordinates (defined in section 1.1) and S and R must satisfy appropriate ordinary differential equations and boundary conditions (given in section 1.2) so that, among other things, when $\text{Im}(\omega) \geq 0$, ψ has finite energy along suitable spacelike hypersurfaces.² Ruling out the exponentially growing mode solutions corresponding to $\text{Im}(\omega) > 0$ is the content of “mode stability.” This was established by Whiting in the ground-breaking [31]. We will extend Whiting’s techniques and establish a *quantitative* understanding of the lack of mode solutions with *real* ω .³ As a byproduct of our methods, we will also be able to simplify the proof of Whiting’s original mode stability result. Next, we will show that this “quantitative mode stability on the real axis” can be upgraded to “integrated local energy decay,” with an explicit constant, for general solutions to the wave equation in any “bounded-frequency regime.” Along the way, we will produce the necessary estimate for section 11.7 of [10].

1.1 The Spacetime

Fix a pair of parameters (a, M) with $|a| < M$, and define

$$r_+ := M + \sqrt{M^2 - a^2}.$$

Define the underlying manifold \mathcal{M} to be covered by a global⁴ “Boyer-Lindquist” coordinate chart

$$(t, r, \theta, \phi) \in \mathbb{R} \times (r_+, \infty) \times \mathbb{S}^2.$$

¹See their discussion in section 11.7 of [10].

²When $\text{Im}(\omega) > 0$ one may take these hypersurfaces to be asymptotically flat. For $\text{Im}(\omega) = 0$ one should instead consider hyperboloidal hypersurfaces terminating on future null infinity.

³See also [15] and [16] which concern solutions to the Cauchy problem of the form $e^{im\phi} \psi_0(t, r, \theta)$ and discuss mode solutions with real ω .

⁴“Global” is to be understood with respect to the usual degeneracy of polar coordinates.

The Kerr metric then takes the form

$$g_{a,M} = - \left(1 - \frac{2Mr}{\rho^2} \right) dt^2 - \frac{4Mar \sin^2 \theta}{\rho^2} dt d\phi + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \sin^2 \theta \frac{\Pi}{\rho^2} d\phi^2, \quad (1.1)$$

$$r_{\pm} := M \pm \sqrt{M^2 - a^2},$$

$$\Delta := r^2 - 2Mr + a^2 = (r - r_+)(r - r_-),$$

$$\rho^2 := r^2 + a^2 \cos^2 \theta,$$

$$\Pi := (r^2 + a^2)^2 - a^2 \sin^2 \theta \Delta.$$

It is convenient to define an $r^*(r) : (r_+, \infty) \rightarrow (-\infty, \infty)$ coordinate up to a constant by

$$\frac{dr^*}{dr} := \frac{r^2 + a^2}{\Delta}.$$

We will often drop the parameters and refer to $g_{a,M}$ as g .

1.2 Separating the Wave Equation: Mode Solutions

When $a = 0$, in addition to possessing the Killing vector field ∂_t , the metric (1.1) is spherically symmetric. Thus, it is immediately clear that the wave equation $\square_{g_{0,M}} \psi = 0$ is separable. When $a \neq 0$ the only Killing vector fields are ∂_t and ∂_ϕ . Nevertheless, as first discovered by Carter [6], the wave equation $\square_g \psi = 0$ remains separable (in an appropriate coordinate system). Indeed, letting $(\omega, m) \in \mathbb{C} \setminus \{0\} \times \mathbb{Z}$, we have

$$\begin{aligned} & \frac{e^{i\omega t} e^{-im\phi}}{\rho^2} \square_g \left(e^{-i\omega t} e^{im\phi} \psi_0(r, \theta) \right) = \\ & \partial_r (\Delta \partial_r) \psi_0 + \left(\frac{(r^2 + a^2)\omega^2 - 4Mar\omega + a^2 m^2}{\Delta} - a^2 \omega^2 \right) \psi_0 + \\ & \frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta) \psi_0 - \left(\frac{m^2}{\sin^2 \theta} - a^2 \omega^2 \cos^2 \theta \right) \psi_0. \end{aligned} \quad (1.2)$$

In fact, the separability of the wave equation follows from the presence on Kerr of a *Killing tensor* [30].

We call

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{dS}{d\theta} \right) - \left(\frac{m^2}{\sin^2 \theta} - a^2 \omega^2 \cos^2 \theta \right) S + \lambda S = 0 \quad (1.3)$$

the “angular ODE.” One can show that when $\omega \in \mathbb{R}$, then (1.3) along with the boundary condition

$$e^{im\phi}S(\theta) \text{ extends smoothly to } \mathbb{S}^2 \quad (1.4)$$

defines a Sturm-Liouville problem with a corresponding collection of eigenfunctions $\{S_{ml}\}_{l=|m|}^\infty$ and real eigenvalues $\{\lambda_{ml}\}_{l=|m|}^\infty$. These $\{S_{ml}\}$ are an orthonormal basis of $L^2(\sin\theta d\theta)$ and are called “oblate spheroidal harmonics.” When $a = 0$ these are simply spherical harmonics, and we label them in the standard way so that $\lambda_{ml} = l(l+1)$. For $a \neq 0$, the labeling is uniquely determined by enforcing continuity in a . Lastly, we note that for ω with sufficiently small imaginary part, one may define the S_{ml} and λ_{ml} via perturbation theory [25].

Now we are ready for the main definition of the section.

Definition 1.1. *Let (\mathcal{M}, g) be a sub-extremal Kerr spacetime with parameters (a, M) . A smooth solution ψ to the wave equation*

$$\square_g \psi = 0 \quad (1.5)$$

is called a “mode solution” if there exist “parameters” $(\omega, m, l) \in \mathbb{C} \setminus \{0\} \times \mathbb{Z} \times \mathbb{Z}$ such that

$$\psi(t, r, \theta, \phi) = e^{-i\omega t} e^{im\phi} S_{ml}(\theta, \omega) R(r, \omega, m, l), \quad (1.6)$$

where

1. S_{ml} satisfies the boundary condition (1.4) and is an eigenfunction with eigenvalue λ_{ml} for the angular ODE (1.3).
2. R is a solution to

$$\partial_r (\Delta \partial_r) R + \left(\frac{(r^2 + a^2)\omega^2 - 4Mamr\omega + a^2m^2}{\Delta} - \lambda_{ml} - a^2\omega^2 \right) R = 0 \quad (1.7)$$

- 3.

$$R \sim (r - r_+) \frac{i(am - 2Mr_+\omega)}{r_+ - r_-} \text{ at } r = r_+.^5 \quad (1.8)$$

⁵This notation means that $R(r)(r - r_+) \frac{-i(am - 2Mr_+\omega)}{r_+ - r_-}$ is smooth at $r = r_+$.

4.

$$R \sim \frac{e^{i\omega r^*}}{r} \text{ at } r = \infty.^6 \quad (1.9)$$

We will often suppress some of the arguments of S_{ml} and R and refer to them as $S_{ml}(\theta)$ and $R(r)$.

Instead of considering $R(r)$, it is often more convenient to work with the function

$$u(r^*) := (r^2 + a^2)^{1/2} R(r).$$

Then, letting primes denote r^* -derivatives, equation (1.7) is equivalent to

$$u'' + (\omega^2 - V) u = 0, \quad (1.10)$$

$$V := \frac{4Mr a m \omega - a^2 m^2 + \Delta(\lambda_{ml} + a^2 \omega^2)}{(r^2 + a^2)^2} + \frac{\Delta}{(r^2 + a^2)^4} (a^2 \Delta + 2Mr(r^2 - a^2)).$$

In appendix A we have collected various facts about the relevant class of ODEs that will be used throughout the paper. The boundary conditions given for R and S_{ml} ((1.8), (1.9), and (1.4)) are uniquely determined by requiring that ψ , given by (1.6), extends smoothly to the horizon,⁷ has finite energy along $\{t = 0\}$ when $\text{Im}(\omega) > 0$, has finite energy along

⁶This notation means that there exists constants $\{C_i\}_{i=0}^\infty$ such that for every $N \geq 1$, $R(r^*) = \frac{e^{i\omega r^*}}{r} \sum_{i=0}^N \frac{C_i}{r^i} + O((r)^{-N-2})$ for large r .

⁷The manifold \mathcal{M} can be extended to a manifold $\tilde{\mathcal{M}}$ such that $\partial\mathcal{M}$ is a null hypersurface called the “horizon.” Since Boyer-Lindquist coordinates would break down at the horizon, one needs a new coordinate system to check whether ψ extends to the horizon. The standard choice is “Kerr-star” coordinates (t^*, r, ϕ^*, θ) :

$$\begin{aligned} \frac{d\bar{t}}{dr} &:= \frac{r^2 + a^2}{\Delta}, \\ \frac{d\bar{\phi}}{dr} &:= \frac{a}{\Delta}, \\ t^*(t, r) &:= t + \bar{t}(r), \\ \phi^*(\phi, r) &:= \phi + \bar{\phi}(r). \end{aligned}$$

In these coordinates the metric becomes

$$\begin{aligned} g = & - \left(1 - \frac{2Mr}{\rho^2}\right) (dt^*)^2 - \frac{4Mar \sin^2 \theta}{\rho^2} dt^* d\phi^* + 2dt^* dr + \\ & \rho^2 d\theta^2 + \sin^2 \theta \frac{\Pi}{\rho^2} (d\phi^*)^2 - 2a \sin^2 \theta dr d\phi^*. \end{aligned}$$

Note that we can now extend the metric to the manifold $\tilde{\mathcal{M}} := (t^*, r, \theta, \phi^*) \in \mathbb{R} \times (0, \infty) \times \mathbb{S}^2$. The horizon is defined to be the null hypersurface $\{r = r_+\}$.

hyperboloidal hypersurfaces when $\text{Im}(\omega) = 0$, and depends smoothly on ω . Furthermore, in section 3.1 we will see these boundary conditions directly arise during the proof of integrated local energy decay. Though we will not study them here, it is worth mentioning that there is a large literature devoted to locating mode solutions with $\text{Im}(\omega) < 0$ (see the review [20]). These are called *quasi-normal modes* and are expected to provide to great deal of dynamical information about the decay of scalar fields. For a sample of the mathematical study of quasi-normal modes and corresponding applications (to black hole spacetimes), we recommend [3], [4], [12], [13], [14], [17], [22], [29], and the references therein.

1.3 Mode Stability Type Statements

Ruling out the exponentially growing mode solutions corresponding to $\text{Im}(\omega) > 0$ is the content of “mode stability (in the upper half plane).” This was established by Whiting in 1989 [31]. However, this turns out not to be the full story. Indeed, the existence of mode solutions with $\omega \in \mathbb{R} \setminus \{0\}$ is a serious obstruction to “integrated local energy decay” for the wave equation. We will call the ruling out of these mode solutions “mode stability on the real axis.” This was first explored numerically in [26]. In addition, [26] presented a heuristic argument (rigorously established in [18]) indicating that mode stability on the real axis would imply mode stability in the upper half plane. In section 3 we will show how one can upgrade mode stability on the real axis to *integrated local energy decay* for the wave equation in any “bounded-frequency regime.” In order for the constant in this estimate to be explicit, however, we will be interested in a quantitative version of mode stability of the real axis.

We turn now to an explanation of “quantitative mode stability.” Observe that if a solution to the angular ODE exists, an asymptotic analysis of (1.10) (see appendix A) allows one to make the following definitions:

Definition 1.2. *Let the parameters $|a| < M$ be fixed. Then define $u_{hor}(r^*, \omega, m, l)$ to be the unique function satisfying*

1. $u_{hor}'' + (\omega^2 - V) u_{hor} = 0.$
2. $u_{hor} \sim (r - r_+) \frac{i(am - 2Mr_+ \omega)}{r_+ - r_-}$ near $r^* = -\infty.$
3. $\left| \left((r(r^*) - r_+) \frac{-i(am - 2Mr_+ \omega)}{r_+ - r_-} u_{hor} \right) (-\infty) \right|^2 = 1.$

Definition 1.3. Let the parameters $|a| < M$ be fixed. Then define $u_{\text{out}}(r^*, \omega, m, l)$ to be the unique function satisfying

1. $u_{\text{out}}'' + (\omega^2 - V) u_{\text{out}} = 0.$
2. $u_{\text{out}} \sim e^{i\omega r^*}$ near $r^* = \infty.$
3. $|(e^{-i\omega r^*} u_{\text{out}})(\infty)|^2 = 1.$

See appendix A for the explicit definition of “ \sim ”. When there is no risk of confusion, we shall drop some or all of u_{hor} ’s and u_{out} ’s arguments. Next, recall that the *Wronskian*

$$u_{\text{out}}'(r^*)u_{\text{hor}}(r^*) - u_{\text{hor}}'(r^*)u_{\text{out}}(r^*)$$

is independent of r^* . Hence, we can define

$$W(\omega, m, l) := u_{\text{out}}'(r^*)u_{\text{hor}}(r^*) - u_{\text{hor}}'(r^*)u_{\text{out}}(r^*). \quad (1.11)$$

This will vanish if and only if u_{out} and u_{hor} are linearly dependent, i.e. there exists a non-trivial solution to (1.10) $\Leftrightarrow W = 0 \Leftrightarrow |W^{-1}| = \infty$. “Quantitative mode stability” consists of producing an upper bound for $|W^{-1}|$ with an explicit dependence on a , M , ω , m , and l .

1.4 Statement of Results

Fix a Kerr spacetime (\mathcal{M}, g) with parameters (a, M) satisfying $|a| < M$, and recall the definition of mode solutions (definition 1.1) and the Wronskian (1.11) given in the previous section.

Our main result about mode solutions is

Theorem 1.4. (*Quantitative Mode Stability on the Real Axis*) Let

$$\mathcal{A} \subset \{(\omega, m, l) \in \mathbb{R} \times \mathbb{Z} \times \mathbb{Z}\}$$

be a set of frequency parameters with

$$C_{\mathcal{A}} := \sup_{(\omega, m, l) \in \mathcal{A}} \left(|\omega| + |\omega|^{-1} + |m| + |l| \right) < \infty.$$

Then

$$\sup_{(\omega, m, l) \in \mathcal{A}} |W^{-1}| \leq G(C_{\mathcal{A}}, a, M)$$

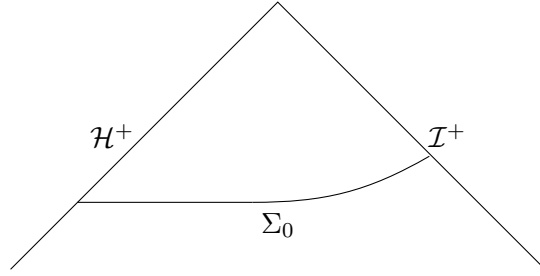
where the function G can, in principle, be given explicitly.

Along the way we will give simple⁸ proofs of

Theorem 1.5. *(Mode Stability)(Whiting [31]) There exist no non-trivial mode solutions corresponding to $\text{Im}(\omega) > 0$.*

Theorem 1.6. *(Mode Stability on the Real Axis) There exist no non-trivial mode solutions corresponding to $\omega \in \mathbb{R} \setminus \{0\}$.*

Before discussing our main application, we need to introduce some more notation. Let ψ be a solution to the wave equation (1.5) on sub-extremal Kerr such that for each fixed Boyer-Lindquist (r, θ, ϕ) , ψ and its coordinate derivatives are square integrable to the future in t . Let Σ_0 be a spacelike hypersurface terminating on the horizon \mathcal{H}^+ and future null infinity which satisfies $t = r^* + O_{r^*}(r^{-1}) + O_{\phi, \theta}(r^{-2})$ as $r \rightarrow \infty$.⁹ The relevant Penrose diagram is given by



Let Σ_1 be the image of Σ_0 under the time 1 map of the flow generated by ∂_t . Define a cutoff χ which is 0 in the past of Σ_0 and identically 1 in the future of Σ_1 . Then define

$$\psi_{\asymp} := \chi\psi.$$

Our application of Theorem 1.4 will be

⁸Using Whiting's integral transformations [31] but avoiding differential transformations or a physical space argument with a new metric.

⁹For an explicit example of such a hypersurface, first define a function $f(r^*)$ by

$$f(r^*) := \int_0^{r^*} \sqrt{1 - \frac{M^2 \Delta}{(r^2 + a^2)^2}} dr$$

Letting α be sufficiently small and β sufficiently large, set

$$\Sigma_0 := \begin{cases} t + f(r^*) = 0 & \text{if } r_+ < r < \alpha \\ t + f(\alpha^*) = 0 & \text{if } \alpha \leq r \leq \beta \\ t - f(r^*) + f(\alpha^*) + f(\beta^*) = 0 & \text{if } \beta < r \end{cases}$$

Theorem 1.7. (*Boundedness of the Microlocal Energy Flux and Integrated Local Energy Decay in the Bounded-Frequency Regime*) Let $\mathcal{B} \subset \mathbb{R}$ and $\mathcal{C} \subset \{(m, l) \in \mathbb{Z} \times \mathbb{Z} : l \geq |m|\}$ be such that

$$C_{\mathcal{B}} := \sup_{\omega \in \mathcal{B}} (|\omega| + |\omega|^{-1}) < \infty$$

$$C_{\mathcal{C}} := \sup_{m, l \in \mathcal{C}} (|m| + |l|) < \infty.$$

Then, for every $r_+ < r_0 < r_1 < \infty$,

$$\begin{aligned} & \int_{\mathcal{H}^+} |P\psi_{\infty}|^2 \\ & + \int_{\mathcal{I}^+} |\partial P\psi_{\infty}|^2 + \int_{\mathbb{R} \times (r_0, r_1) \times \mathbb{S}^2} |\partial P\psi_{\infty}|^2 \leq \\ & B(r_0, r_1, C_{\mathcal{B}}, C_{\mathcal{C}}, a, M) \int_{\Sigma_0} |\partial\psi|^2 \end{aligned} \tag{1.12}$$

where $|\partial\psi|^2$ denotes a term proportional to a non-degenerate energy flux of a globally timelike vector field, P denotes a projection in phase space to frequencies supported in \mathcal{B} and \mathcal{C} , i.e.

$$P\psi_{\infty}(t, r, \theta, \phi) :=$$

$$\int_{\mathcal{B}} \sum_{(m, l) \in \mathcal{C}} \left(\int_0^\pi \int_0^{2\pi} \int_{-\infty}^\infty \psi_{\infty} e^{i\omega\tau} e^{-im\varphi} S_{ml} \sin\vartheta \, d\tau \, d\varphi \, d\vartheta \right) S_{ml} e^{im\phi} e^{-i\omega t} \, d\omega,$$

and the integrals are with respect to the induced volume forms. In coordinates

$$|\partial\psi|_{\Sigma_0}^2 \approx ((\partial_t + \partial_{r^*})\psi)^2 + r^{-2}((\partial_t - \partial_{r^*})\psi)^2 + r^{-2}((\partial_\phi\psi)^2 + (\partial_\theta\psi)^2),$$

$$|\partial P\psi_{\infty}|_{\mathcal{I}^+}^2 \approx \lim_{r \rightarrow \infty} r^2 \left(|\partial_t P\psi_{\infty}|^2 + |\partial_r P\psi_{\infty}|^2 + r^{-2}((\partial_\phi P\psi_{\infty})^2 + (\partial_\theta P\psi_{\infty})^2) \right).$$

Note that the spacetime volume form satisfies

$$dVol_{(t, r, \theta, \phi)} \approx r^2 \sin\theta \, dt \, dr \, d\theta \, d\phi.$$

The function $B(r_0, r_1, C_{\mathcal{B}}, C_{\mathcal{C}}, a, M)$ can, in principle, be given explicitly.

Of course, since we are in a bounded-frequency regime, the zeroth order estimate along the horizon (1.12) controls the microlocal energy flux along the horizon:

$$\int_{\mathcal{B}} \sum_{(m,l) \in \mathcal{C}} \omega (am - 2Mr_+ \omega) |u(-\infty)|^2 d\omega.$$

Here

$$u(r^*) := (r^2 + a^2)^{1/2} R(r)$$

where $R(r, \omega, m, l)$ is the projection of the Fourier transform in t of ψ_{∞} onto the oblate spheroidal harmonics S_{ml} , i.e.

$$R(r) := \int_0^\pi \int_0^{2\pi} \int_{-\infty}^\infty \psi_{\infty} e^{i\omega t} e^{-im\phi} S_{ml} \sin \theta dt d\phi d\theta.$$

The estimate for this term is utilized in Dafermos' and Rodnianski's proof of integrated local energy decay for the wave equation [10]. For this application, it is very important that the right hand side is at the level of energy.

Before diving into the proofs of our results, we will review the case of mode solutions on Schwarzschild ($a = 0$) and what is already known about mode solutions on Kerr.

1.5 Modes on Schwarzschild

It is instructive to observe that the counterpart to mode stability in the Riemannian setting¹⁰ is the “automatic” fact that the Laplace-Beltrami operator has no spectrum in the upper half plane. A better way to see the triviality of Riemannian mode stability is to note that the existence of a uniformly timelike vector field ∂_t immediately implies the uniform boundedness of a non-degenerate energy [1].

Recall that the Schwarzschild spacetime is the Kerr spacetime with vanishing angular momentum ($a = 0$). This is not a product metric; nevertheless, ∂_t is a timelike Killing vector field for all $r > r_+$, the associated conserved energy is coercive, and mode stability is immediately established in a similar fashion to the previous paragraph.¹¹

Mode stability on the real axis for Schwarzschild is more subtle since real mode solutions have infinite energy along asymptotically flat hypersurfaces. However, this does not preclude physical space methods; one simply observes

¹⁰This is the case of a product metric $(\mathbb{R} \times N, -dt^2 + g_N)$ with (N, g_N) complete and Riemannian.

¹¹Of course, ∂_t becomes null on the horizon, and thus the conserved energy degenerates as $r \rightarrow r_+$. However, a moment's thought shows that this does not affect the argument.

1. The boundary condition at infinity implies that real mode solutions have finite energy along the hypersurface Σ_0 .
2. A straightforward computation shows that the energy flux for such real modes along the portion of null infinity in the future of Σ_0 must be infinite.
3. The energy identity associated to ∂_t implies that the energy flux along the portion of null infinity in the future of Σ_0 must be less than or equal to the energy flux along Σ_0 .

This is a clear contradiction to the existence of real modes.

For later purposes it will be convenient to revisit these arguments from a “microlocal” point of view. In phase space, the analogue of the energy flux is the microlocal energy current:

$$Q_T(r^*) := \text{Im} \left(u' \overline{\omega u} \right).$$

Let us show how the microlocal energy can be used to give a short proof of mode stability. Suppose we have a mode solution with corresponding $u(r^*)$ and $\omega = \omega_R + i\omega_I$ for some $\omega_I > 0$. First, we observe that the boundary conditions (1.8) and (1.9) imply that $Q_T(\pm\infty) = 0$. Next, we compute

$$\begin{aligned} -(Q_T)' &= \omega_I |u'|^2 + \text{Im} \left((\omega^2 - V) \overline{\omega} \right) |u|^2 = \\ &\omega_I \left(|u'|^2 + \left(|\omega|^2 + \frac{(r-2M)(rl(l+1)+2M)}{r^4} \right) |u|^2 \right). \end{aligned}$$

Since the coefficients of $|u'|^2$ and $|u|^2$ are positive, the fundamental theorem of calculus implies that u is identically 0. Algebraically, we are exploiting the fact that the potential V does not depend on ω and is positive.

Now consider a real mode solution with corresponding $u(r^*)$ and $\omega \in \mathbb{R} \setminus \{0\}$. This time we have “conservation of energy,”

$$(Q_T)' = 0.$$

Integrating gives

$$\begin{aligned} Q_T(\infty) - Q_T(-\infty) &= 0 \Rightarrow \\ \omega^2 |u(\infty)|^2 + 2Mr_+ \omega^2 |u(-\infty)|^2 &= 0. \end{aligned}$$

We have used the boundary conditions (1.8) and (1.9) to evaluate the microlocal energy current at $\pm\infty$. It immediately follows that u vanishes identically.

1.6 Modes on Kerr: The Ergoregion, Superradiance, and Whiting's Transformations

On the Kerr spacetime all of these arguments break down.

In the ergoregion

$$\Delta - a^2 \sin^2 \theta < 0$$

the Killing vector field ∂_t is no longer timelike. Hence, the associated conserved quantity is no longer coercive and is useless by itself.

At the level of the ODE, we may again define a microlocal energy current:

$$Q_T := \text{Im} (u' \overline{u}).$$

However,

$$\begin{aligned} \text{Im} ((\omega^2 - V) \overline{u}) = \\ \omega_I \left(|\omega|^2 - \frac{a^2 m^2}{(r^2 + a^2)^2} + \frac{\Delta}{(r^2 + a^2)^4} (a^2 \Delta + 2Mr(r^2 - a^2)) \right) + \\ \frac{\Delta}{(r^2 + a^2)^2} \text{Im} ((\lambda_{ml} + a^2 \omega^2) \overline{u}) \end{aligned}$$

is no longer always positive. In fact, for $\omega_I > 0$

$$\text{Im} ((\omega^2 - V) \overline{u}) (-\infty) = \omega_I \left(|\omega|^2 - \frac{a^2 m^2}{4M^2 r_+^2} \right) < 0 \Leftrightarrow$$

$$|am| - 2Mr_+ |\omega| > 0.$$

This troublesome frequency regime also arises if $\omega \in \mathbb{R} \setminus \{0\}$. For such ω we still have “conservation of energy,”

$$(Q_T)' = 0.$$

Integrating and evaluating with the boundary conditions (1.8) and (1.9) gives

Proposition 1.8. *(The Microlocal Energy Estimate)*

$$\omega^2 |u(\infty)|^2 - \omega (am - 2Mr_+ \omega) |u(-\infty)|^2 = 0.$$

If $\omega (am - 2Mr_+ \omega) < 0$, then this gives a successful estimate of the boundary terms $|u(-\infty)|^2$ and $|u(\infty)|^2$. However, if

$$\omega (am - 2Mr_+ \omega) \geq 0, \tag{1.13}$$

then proposition 1.8 fails to give an estimate for $|u(-\infty)|^2$ and $|u(\infty)|^2$. In the case of (1.13) we say that our frequency parameters are *superradiant*. The existence of superradiant frequencies is the phase space manifestation of the fact that the physical space energy flux associated to ∂_t may be negative along the horizon, i.e. energy can be extracted from a spinning black hole.

Despite these difficulties, in [31] Whiting was able to give a relatively short proof of mode stability for a wide class of equations on sub-extremal Kerr, including the wave equation $\square_g \psi = 0$, i.e. Theorem 1.5. By closely examining the structure of u 's and S_{ml} 's equations, Whiting found (appropriately non-degenerate) integral and differential transformations taking u to \tilde{u} and S_{ml} to \tilde{S}_{ml} such that

$$\tilde{\psi}(t, r, \theta, \phi) := (r^2 + a^2)^{-1/2} e^{-i\omega t} e^{im\phi} \tilde{S}_{ml}(\theta) \tilde{u}(r^*(r))$$

satisfied a wave equation $\square_{\tilde{g}} \tilde{\psi} = 0$ associated to a new metric \tilde{g} for which there was no ergoregion. After this miracle, the proof concluded with a physical space energy argument as in our discussion of Schwarzschild in section 1.5.

2 The Wronskian Estimate and Proofs of Mode Stability

In this section we will explain our extension of Whiting's integral transformations and use this to prove Theorems 1.4, 1.5, and 1.6.

It turns out to be useful to work with the inhomogeneous version of R 's and u 's equations:

$$\Delta \frac{d}{dr} \left(\Delta \frac{dR}{dr} \right) - \tilde{V} R = \Delta(r^2 + a^2) F(r) =: \Delta \hat{F}, \quad (2.1)$$

$$\tilde{V} := -(r^2 + a^2)^2 \omega^2 + 4Mamr\omega - a^2 m^2 + \Delta(\lambda_{ml} + a^2 \omega^2).$$

Recalling that $u(r^*) = (r^2 + a^2)^{-1/2} R(r)$, we have

$$u'' + (\omega^2 - V) u = H, \quad (2.2)$$

$$V := \frac{4Mram\omega - a^2 m^2 + \Delta(\lambda_{ml} + a^2 \omega^2)}{(r^2 + a^2)^2} + \frac{\Delta}{(r^2 + a^2)^4} (a^2 \Delta + 2Mr(r^2 - a^2)),$$

$$H(r^*) := \frac{\Delta}{(r^2 + a^2)^{1/2}} F(r). \quad (2.3)$$

Our starting point is Whiting's integral transformation:

$$\tilde{u}(x^*) := (x^2 + a^2)^{1/2} (x - r_+)^{-2iM\omega} e^{-i\omega x} \times \quad (2.4)$$

$$\int_{r_+}^{\infty} e^{\frac{2i\omega}{r_+ - r_-} (x - r_-)(r - r_-)} (r - r_-)^{\eta} (r - r_+)^{\xi} e^{-i\omega r} R(r) dr. \quad (2.5)$$

Here η and ξ are given by

$$\eta := \frac{-i(am - 2Mr_- \omega)}{r_+ - r_-},$$

$$\xi := \frac{i(am - 2Mr_+ \omega)}{r_+ - r_-}.$$

In [31] Whiting used the above transformation only on modes satisfying the homogeneous equation with $\text{Im}(\omega) > 0$, and the integral was thus absolutely convergent. Since we shall also allow $\omega \in \mathbb{R} \setminus \{0\}$, at first, \tilde{u} only makes sense as an L^2_{loc} function. Nevertheless, in section 4 we will establish

Proposition 2.1. *Let $\text{Im}(\omega) \geq 0$, $\omega \neq 0$, R solve the inhomogeneous radial ODE (2.1), and R satisfy the boundary conditions from definition 1.1. Define \tilde{u} via Whiting's integral transformation (2.4). Then $\tilde{u}(x)$ is in $H^1_{\text{loc}}(r_+, \infty)$ and, letting primes denote x^* -derivatives, satisfies*

$$\tilde{u}'' + \Phi \tilde{u} = \tilde{H},$$

where

$$\tilde{H}(x^*) := \frac{(x - r_+)(x - r_-)}{(x^2 + a^2)^2} \tilde{G}(x), \quad (2.6)$$

$$\tilde{G}(x) := (x^2 + a^2)^{1/2} (x - r_+)^{-2iM\omega} e^{-i\omega x} \times \int_{r_+}^{\infty} e^{\frac{2i\omega}{r_+ - r_-} (x - r_-)(r - r_-)} (r - r_-)^{\eta} (r - r_+)^{\xi} e^{-i\omega r} \hat{F}(r) dr,$$

$$\Phi(x^*) := \frac{(x - r_-)\tilde{\Phi}(x)}{(x^2 + a^2)^2} - \frac{(x - r_+)(x - r_-)}{(x^2 + a^2)^4} (a^2(x - r_+)(x - r_-) + 2Mx(x^2 - a^2)),^{12}$$

$$\begin{aligned} \tilde{\Phi}(x) := & \omega^2(x - r_+)^2(x - r_-) - \left(4M\omega^2 + \frac{4\omega(am - 2Mr_+\omega)}{r_+ - r_-} \right) (x - r_-)(x - r_+) + \\ & 4M^2\omega^2(x - r_-) + (2am\omega - \lambda_{ml} - a^2\omega^2)(x - r_+). \end{aligned}$$

¹²For mode stability on the real axis, it is only important that Φ is real.

Of course, it is important to understand the boundary conditions for \tilde{u} . When $\text{Im}(\omega) > 0$, the following quite crude analysis of \tilde{u} is sufficient.

Proposition 2.2. *If $\text{Im}(\omega) > 0$, then*

1. $\tilde{u} = O\left((x - r_+)^{2M\text{Im}(\omega)}\right)$ as $x \rightarrow r_+$.
2. $\tilde{u}' = O\left((x - r_+)^{2M\text{Im}(\omega)}\right)$ as $x \rightarrow r_+$.
3. $\tilde{u} = O\left(e^{-\text{Im}(\omega)x} x^{1+2M\text{Im}(\omega)}\right)$ as $x \rightarrow \infty$.
4. $\tilde{u}' = O\left(e^{-\text{Im}(\omega)x} x^{1+2M\text{Im}(\omega)}\right)$ as $x \rightarrow \infty$.

When $\omega \in \mathbb{R} \setminus \{0\}$ we need to be a little more precise.

Proposition 2.3. *If $\omega \in \mathbb{R} \setminus \{0\}$, then*

1. \tilde{u} is uniformly bounded.
2. $|\tilde{u}(\infty)|^2 = \frac{(r_+ - r_-)^2}{4M\omega^2 r_+} |u(-\infty)|^2$.
3. \tilde{u}' is uniformly bounded.
4. $\tilde{u}' - i\omega\tilde{u} = O(x^{-1})$ at $x^* = \infty$.
5. $\tilde{u}' + \frac{i\omega(r_+ - r_-)}{r_+}\tilde{u} = O(x - r_+)$ at $x^* = -\infty$.

Let's see how these propositions restricted to the homogeneous case allow for immediate proofs of both mode stability in the upper half plane and on the real axis via the microlocal energy current:

$$\tilde{Q}_T := \text{Im}(\tilde{u}'\overline{\omega\tilde{u}}).$$

Proof. (Mode Stability, Theorem 1.5) Suppose we had a mode solution with corresponding $(u, S_{ml}, \lambda_{ml})$ and $\omega = \omega_R + i\omega_I$ with $\omega_I > 0$. Let \tilde{u} be defined by (2.4). Proposition 2.2 implies that $\tilde{Q}_T(\pm\infty) = 0$. We proceed as in our discussion of Schwarzschild from section 1.5 with \tilde{u} replacing u :

$$0 = -\tilde{Q}_T|_{-\infty}^{\infty} = -\int_{-\infty}^{\infty} (\tilde{Q}_T)' dr^* = \int_{-\infty}^{\infty} \left(\omega_I |\tilde{u}'|^2 + \text{Im}(\Phi\overline{\omega}) |\tilde{u}|^2 \right) dr^*.$$

Hence, if we can show that $\text{Im}(\Phi\overline{\omega}) \geq 0$, we may conclude that \tilde{u} vanishes.

An easy computation using the formula from proposition 2.1 gives

$$\text{Im}(\Phi\overline{\omega}) = \omega_I \left(\frac{(x - r_-)}{(x^2 + a^2)^2} \Psi_0 + \frac{(x - r_+)(x - r_-)}{(x^2 + a^2)^4} \Psi_1 \right) -$$

$$\frac{(x-r_+)(x-r_-)}{(x^2+a^2)^2} \operatorname{Im}((\lambda_{ml} + a^2\omega^2)\bar{\omega}),$$

$$\Psi_0 := |\omega|^2 (x-r_+)^2(x-r_-) + \frac{8M^2|\omega|^2}{r_+-r_-}(x-r_-)(x-r_+) + 4M^2|\omega|^2(x-r_-),$$

$$\Psi_1 := a^2(x-r_+)(x-r_-) + 2Mx(x^2-a^2).$$

All of these terms are clearly positive except for $-\operatorname{Im}((\lambda_{ml} + a^2\omega^2)\bar{\omega})$. For this term we need to return to S_{ml} 's equation (1.3):

$$\frac{1}{\sin\theta} \frac{d}{d\theta} \left(\sin\theta \frac{dS_{ml}}{d\theta} \right) - \left(\frac{m^2}{\sin^2\theta} + a^2\omega^2 \sin^2\theta \right) S_{ml} + (\lambda_{ml} + a^2\omega^2) S_{ml} = 0.$$

Now multiply the equation by $\overline{\omega S_{ml}} \sin\theta$, integrate by parts, and take the imaginary part. There are no boundary terms due to S_{ml} 's boundary conditions,¹³ and we find

$$\begin{aligned} \omega_I \int_0^\pi \left(\left| \frac{dS_{ml}}{d\theta} \right|^2 + \left(\frac{m^2}{\sin^2\theta} + a^2|\omega|^2 \sin^2\theta \right) |S_{ml}|^2 \right) \sin\theta d\theta = \\ - \int_0^\pi (\operatorname{Im}((\lambda_{ml} + a^2\omega^2)\bar{\omega})) |S_{ml}|^2 \sin\theta d\theta \Rightarrow \\ -\operatorname{Im}((\lambda_{ml} + a^2\omega^2)\bar{\omega}) \geq 0. \end{aligned}$$

We conclude that $\operatorname{Im}(\Phi\bar{\omega})$ is positive, and hence that \tilde{u} must vanish.

In terms of R , this implies that

$$\tilde{R}(x) := \int_{r_+}^\infty e^{\frac{2i\omega}{r_+-r_-}(x-r_-)(r-r_-)} (r-r_-)^\eta (r-r_+)^\xi e^{-i\omega r} R(r) dr$$

vanishes for all $x \in (r_+, \infty)$. To see that this implies that R vanishes, we first extend R by 0 to all of \mathbb{R} and note that the Fourier transform of $(r-r_-)^\eta (r-r_+)^\xi e^{-i\omega r} R(r)$ is, up to a change of variables,

$$\hat{R}(z) := \int_{-\infty}^\infty e^{2i|\omega|^2 z(r-r_-)} (r-r_-)^\eta (r-r_+)^\xi e^{-i\omega r} R(r) dr.$$

In view of the support of R , \hat{R} extends to a holomorphic function on the upper half plane. The vanishing of \hat{R} for $x \in (r_+, \infty)$ implies that \hat{R} vanishes along the line $\{\frac{y}{\omega} : y \in (1, \infty)\}$. Analyticity implies that \hat{R} and hence R itself vanishes. \square

¹³Recall that the boundary conditions (1.4) required that $e^{im\phi} S_{ml}(\theta)$ extend smoothly to \mathbb{S}^2 . More explicitly, let $x := \cos\theta$; then an asymptotic analysis of the angular ODE shows that the boundary condition (1.4) is equivalent to $S_{ml} \sim (x \pm 1)^{|m|}$ as $x \rightarrow \mp 1$.

Note that the above proof occurs completely at the level of \tilde{u} and S_{ml} . In particular, we neither need Whiting's differential transformations of S_{ml} (see section IV of [31]) nor a physical space argument with a new metric (see section VI of [31]).

Proof. (Mode Stability on the Real Axis, Theorem 1.6) Suppose we have a mode solution with corresponding $(u, S_{ml}, \lambda_{ml})$ and $\omega \in \mathbb{R} \setminus \{0\}$. Let \tilde{u} be defined by (2.4). Then, noting that Φ from proposition 2.1 is real, we have conservation of energy:

$$\left(\tilde{Q}_T\right)' = 0 \Rightarrow$$

$$\tilde{Q}_T(\infty) - \tilde{Q}_T(-\infty) = 0.$$

Now the boundary conditions from proposition 2.3 imply that we get a useful estimate out of this:

$$\begin{aligned} & \tilde{Q}_T(\infty) - \tilde{Q}_T(-\infty) = \\ & \frac{1}{2} \left(\omega^2 |\tilde{u}(\infty)|^2 + |\tilde{u}'(\infty)|^2 + \omega^2 \frac{r_+ - r_-}{r_+} |\tilde{u}(-\infty)|^2 + \frac{r_+}{r_+ - r_-} |\tilde{u}'(-\infty)|^2 \right). \end{aligned}$$

We conclude that \tilde{u} must vanish.

In terms of R , we see that

$$\tilde{R}(y) := \int_{-\infty}^{\infty} e^{2i\omega y(r-r_-)} (r-r_-)^{\eta} (r-r_+)^{\xi} e^{-i\omega r} R(r) dr$$

vanishes for $y \in (1, \infty)$, where we have extended R by 0 so that it is defined on all of \mathbb{R} . However, it is well known that the Fourier transform of a non-trivial function supported in $(0, \infty)$ cannot vanish on an open set.¹⁴ As an alternative to this unique continuation argument, one may instead use the fact from proposition 2.3 that

$$|\tilde{u}(\infty)|^2 = \frac{(r_+ - r_-)^2}{4M\omega^2 r_+} |u(-\infty)|^2$$

to conclude that $u(-\infty)$ and hence u vanishes. \square

Note that this proof is even simpler than the proof of mode stability in the upper half plane since we only need to refer to \tilde{u} .

Let's now discuss the proof of Theorem 1.4. To produce quantitative estimates for the Wronskian we shall need to work a little harder than we

¹⁴This follows from holomorphically extending to the upper half plane and the Schwarz reflection principle.

did for the qualitative statements. Let \mathcal{A} be as in the statement 1.4, let $(\omega, m, l) \in \mathcal{A}$, and u solve (2.2) with non-zero right hand side (2.3). Define \tilde{u} and \tilde{H} via (2.4) and (2.6). We have

$$\begin{aligned} \left(\tilde{Q}_T\right)' &= \omega \operatorname{Im} \left(\tilde{H} \tilde{u}\right) \Rightarrow \\ \tilde{Q}_T(\infty) - \tilde{Q}_T(-\infty) &= \omega \int_{-\infty}^{\infty} \operatorname{Im} \left(\tilde{H} \tilde{u}\right) dx^*. \end{aligned}$$

As above, the boundary conditions from proposition 2.3 imply that we get a useful estimate:

$$\begin{aligned} \tilde{Q}_T(\infty) - \tilde{Q}_T(-\infty) &= \\ \frac{1}{2} \left(\omega^2 |\tilde{u}(\infty)|^2 + |\tilde{u}'(\infty)|^2 + \omega^2 \frac{r_+ - r_-}{r_+} |\tilde{u}(-\infty)|^2 + \frac{r_+}{r_+ - r_-} |\tilde{u}'(-\infty)|^2 \right). \end{aligned}$$

For any $\epsilon > 0$ changing variables and applying Plancherel implies

$$\omega \int_{-\infty}^{\infty} \operatorname{Im} \left(\tilde{H} \tilde{u}\right) dr^* \lesssim (4\epsilon)^{-1} \int_{r_+}^{\infty} |F(r)|^2 r^4 dr + \epsilon \int_{r_+}^{\infty} |R(r)|^2 dr.$$

From proposition 2.3 we have

$$|\tilde{u}(\infty)|^2 = \frac{(r_+ - r_-)^2}{4M\omega^2 r_+} |u(-\infty)|^2.$$

We conclude

Proposition 2.4. *For $(\omega, m, l) \in \mathcal{A}$ and u solving satisfying (2.2) with right hand side (2.3), we have*

$$|u(-\infty)|^2 \lesssim (4\epsilon)^{-1} \int_{r_+}^{\infty} |F(r)|^2 r^4 dr + \epsilon \int_{r_+}^{\infty} |R(r)|^2 dr.$$

In section 5 we will show that this control of $|u(-\infty)|^2$ is sufficient to work directly with u 's/ R 's ODE and estimate

$$\int_{r_+}^{\infty} |R(r)|^2 dr.$$

Consequently, we will obtain

Proposition 2.5. *For $(\omega, m, l) \in \mathcal{A}$ and u solving satisfying (2.2) with right hand side (2.3), we have*

$$|u(-\infty)|^2 \lesssim \int_{r_+}^{\infty} |F(r)|^2 r^4 dr.$$

It is important to observe that there are too many powers of r on the right hand side for the above lemma to be directly useful for Theorem 1.7. Instead, we will directly construct solutions to the inhomogeneous radial ODE via the following lemma.

Lemma 2.6. *Let $H(x^*)$ be compactly supported. For any $(\omega, m, l) \in \mathcal{A}$, define*

$$u(r^*) := W^{-1} \left(u_{\text{out}}(r^*) \int_{-\infty}^{r^*} u_{\text{hor}}(x^*) H(x^*) dx^* + u_{\text{hor}}(r^*) \int_{r^*}^{\infty} u_{\text{out}}(x^*) H(x^*) dx^* \right).$$

Then

$$u'' + (\omega^2 - V) u = H$$

and u satisfies the boundary conditions of a mode solution (1.8) and (1.9).

Proof. This is a simple computation. \square

Then, for H is compactly supported, lemma 2.6 gives

$$|u(-\infty)|^2 = |W|^{-2} \left| \int_{-\infty}^{\infty} u_{\text{out}}(x^*) H(x^*) dx^* \right|^2.$$

Combining this with proposition 2.5 gives

$$|W|^{-2} \lesssim \frac{\int_{r_+}^{\infty} |F(r)|^2 r^4 dr}{\left| \int_{-\infty}^{\infty} u_{\text{out}}(x^*) H(x^*) dx^* \right|^2} = \frac{\int_{r_+}^{\infty} |F(r)|^2 r^4 dr}{\left| \int_{r_+}^{\infty} u_{\text{out}}(r^*(r)) F(r) (r^2 + a^2)^{1/2} dr \right|^2}.$$

Of course, W is independent of F , so it remains to pick any particular compactly supported F we want so that the right hand side is finite. Since for sufficiently large x , $|u_{\text{out}} - e^{i\omega x^*}| \leq \frac{C}{x}$ for an explicit constant C (appendix A), it is certainly possible to find such an F . Thus, we have produced a quantitative bound for W^{-1} .

3 Proof of the Energy Flux Bound and Integrated Local Energy Decay

In this section we shall show that Theorem 1.4 (quantitative mode stability on the real axis) implies Theorem 1.7 (boundedness of the energy flux and integrated local energy decay in the bounded-frequency regime).

3.1 Some Exponential Damping and Boundary Conditions

We shall use the notation introduced for the statement of Theorem 1.7. In order to avoid dealing with certain technical issues near null infinity, it turns out to be easier for the proof to work with

$$\psi_\epsilon := e^{-\epsilon t} \psi \text{ for } \epsilon > 0.$$

Recall that before the statement of Theorem 1.7 we defined a cutoff χ such that χ is 0 in the past of Σ_0 and identically 1 in the future of Σ_1 . We then define

$$\begin{aligned} \psi_{\epsilon, \infty} &:= \chi \psi_\epsilon, \\ E_\epsilon &:= e^{-\epsilon t} ((\square_g \chi) \psi + 2\nabla^\mu \chi \nabla_\mu \psi), \\ \omega_\epsilon &:= \omega + i\epsilon. \end{aligned}$$

Next, we let F_ϵ be the projection onto the oblate spheroidal harmonics of the Fourier transform of $(r^2 + a^2)^{-1} \rho^2 E_\epsilon$, $u_\epsilon(r^*)$ be the projection onto the oblate spheroidal harmonics of the Fourier transform of $(r^2 + a^2)^{1/2} \psi_\epsilon$, and

$$H_\epsilon(r^*) = \frac{\Delta}{(r^2 + a^2)^{1/2}} F_\epsilon.$$

We get

$$\begin{aligned} u_\epsilon'' + (\omega_\epsilon^2 - V_\epsilon) u_\epsilon &= H_\epsilon, \\ V &:= \frac{4Mram\omega_\epsilon - a^2m^2 + \Delta(\lambda_{ml} + a^2\omega_\epsilon^2)}{(r^2 + a^2)^2} + \frac{\Delta}{(r^2 + a^2)^4} (a^2\Delta + 2Mr(r^2 - a^2)). \end{aligned} \tag{3.1}$$

For notational ease, we shall introduce one last set of definitions. Recalling the notations established in definition 1.2 and 1.3, we set

$$\begin{aligned} u_{\text{hor}, \epsilon}(r^*) &:= u_{\text{hor}}(r^*, \omega_\epsilon, m, l), \\ u_{\text{out}, \epsilon}(r^*) &:= u_{\text{out}}(r^*, \omega_\epsilon, m, l), \\ W_\epsilon &:= u'_{\text{out}, \epsilon} u_{\text{hor}, \epsilon} - u'_{\text{hor}, \epsilon} u_{\text{out}, \epsilon}. \end{aligned}$$

These will satisfy

1. $u_{\text{hor}, \epsilon}'' + (\omega_\epsilon^2 - V_\epsilon) u_{\text{hor}, \epsilon} = 0.$
2. $u_{\text{hor}, \epsilon} \sim (r - r_+) \frac{i(am - 2Mr_+ \omega_\epsilon)}{r_+ - r_-}$ near $r^* = -\infty.$
3. $\left| \left((r(\cdot) - r_+) \frac{-i(am - 2Mr_+ \omega_\epsilon)}{r_+ - r_-} u_{\text{hor}, \epsilon} \right) (-\infty) \right|^2 = 1.$

4. $u_{\text{out},\epsilon}'' + (\omega_\epsilon^2 - V_\epsilon) u_{\text{out},\epsilon} = 0$.
5. $u_{\text{out},\epsilon} \sim e^{i\omega_\epsilon r^*}$ near $r^* = \infty$
6. $|(e^{-i\omega_\epsilon(\cdot)} u_{\text{out},\epsilon})(-\infty)|^2 = 1$.
7. $W_\epsilon \neq 0$ by mode stability.

The key result of this section is

Proposition 3.1.

$$u_\epsilon(r^*) = W_\epsilon^{-1} \left(u_{\text{out},\epsilon}(r^*) \int_{-\infty}^{r^*} u_{\text{hor},\epsilon}(x^*) H_\epsilon(x^*) dx^* + u_{\text{hor},\epsilon}(r^*) \int_{r^*}^{\infty} u_{\text{out},\epsilon}(x^*) H_\epsilon(x^*) dx^* \right).$$

Proof. We start by observing that as a consequence of assuming that ψ and its coordinate derivatives are square integrable to the future in time, one may use a straightforward energy estimate to easily see that $\chi\psi$ is uniformly bounded. Along the support of ψ_ϵ , there exists a constant B such that $t \geq |r^*| - B$. We conclude that ψ_ϵ and hence u_ϵ and H_ϵ are exponentially decreasing as $r^* \rightarrow \pm\infty$. Hence, we can define

$$\hat{u}_\epsilon(r^*) := W_\epsilon^{-1} \left(u_{\text{out},\epsilon}(r^*) \int_{-\infty}^{r^*} u_{\text{hor},\epsilon}(x^*) H_\epsilon(x^*) dx^* + u_{\text{hor},\epsilon}(r^*) \int_{r^*}^{\infty} u_{\text{out},\epsilon}(x^*) H_\epsilon(x^*) dx^* \right).$$

Now, a simple computation shows that

$$(\hat{u}_\epsilon - u_\epsilon)'' + (\omega_\epsilon^2 - V) (\hat{u}_\epsilon - u_\epsilon) = 0.$$

Furthermore, $\hat{u}_\epsilon - u_\epsilon$ is exponentially decreasing as $r^* \rightarrow \pm\infty$. From ODE theory (appendix A), $\hat{u}_\epsilon - u_\epsilon$ must be asymptotic to a linear combination of

$$\left\{ (r - r_+)^{\frac{i(am-2Mr_+\omega_\epsilon)}{r_+-r_-}}, (r - r_+)^{\frac{-i(am-2Mr_+\omega_\epsilon)}{r_+-r_-}} \right\}.$$

The only possible choice is

$$\hat{u}_\epsilon - u_\epsilon \sim (r - r_+)^{\frac{i(am-2Mr_+\omega_\epsilon)}{r_+-r_-}} \text{ at } r^* = -\infty.$$

Next, ODE theory (appendix A) implies that near infinity, $\hat{u}_\epsilon - u_\epsilon$ must be asymptotic to a linear combination of

$$\left\{ e^{i\omega_\epsilon r^*}, e^{-i\omega_\epsilon r^*} \right\}.$$

The exponential decay of $\hat{u}_\epsilon - u_\epsilon$ singles out

$$\hat{u}_\epsilon - u_\epsilon \sim e^{i\omega_\epsilon r^*} \text{ at } r^* = \infty.$$

Thus, $\hat{u}_\epsilon - u_\epsilon$ satisfies the boundary conditions of a mode solution. Finally, mode stability in the upper half plane implies that $\hat{u} = u_\epsilon$. \square

3.2 The Estimate

We keep the notation introduced in the previous section. Also, recall the definition of $|\partial\psi|^2$ given in the statement of Theorem 1.7. We start with a lemma.

Lemma 3.2.

$$\limsup_{\epsilon \rightarrow 0} \int_{\mathcal{B}} \sum_{m,l} \int_{r_+}^{\infty} |F_\epsilon|^2 r^2 dr d\omega \lesssim \int_{\Sigma_0} |\partial\psi|^2$$

In particular, even though there are 0th order terms in F , there are only derivatives of ψ on the right hand side.¹⁵

Proof. By Plancherel,

$$\begin{aligned} \limsup_{\epsilon \rightarrow 0} \int_{\mathcal{B}} \sum_{m,l} \int_{r_+}^{\infty} |F_\epsilon|^2 r^2 dr d\omega &\lesssim \\ \limsup_{\epsilon \rightarrow 0} \int_{(t,r,\theta,\phi)} \left(|(\square_g \chi) \psi_\epsilon|^2 + |\nabla^\mu \chi \nabla_\mu \psi_\epsilon|^2 \right) r^2 \sin \theta dt dr d\theta d\phi. \end{aligned}$$

We will consider the two terms on the right hand side separately.

For the second term we simply observe that the asymptotic behavior of Σ_0 implies

$$\begin{aligned} |\nabla^\mu \chi \nabla_\mu \psi_\epsilon|^2 &\lesssim \\ 1_{\text{supp}(\nabla \chi)} \left(|(\partial_t + \partial_{r^*}) \psi_\epsilon|^2 + O(r^{-2}) |(\partial_t - \partial_{r^*}) \psi_\epsilon|^2 + O(r^{-4}) \left(|\partial_\theta \psi_\epsilon|^2 + |\partial_\phi \psi_\epsilon|^2 \right) \right) \end{aligned}$$

where $1_{\text{supp}(\nabla \chi)}$ denotes the indicator function on the support of $\nabla \chi$.

For the first term, first pick a null frame $(L, \underline{L}, E_1, E_2)$ where

$$g(L, L) = g(\underline{L}, \underline{L}) = g(E_1, E_2) = g(L, E_i) = g(\underline{L}, E_i) = 0,$$

$$g(L, \underline{L}) = -2,$$

¹⁵If Σ_0 were asymptotically flat, 0th order terms would have to be present on the right hand side.

$$g(E_1, E_1) = g(E_2, E_2) = 1,$$

$$L = \partial_t + \partial_{r^*} + O(r^{-1}),$$

$$\underline{L} = \partial_t - \partial_{r^*} + O(r^{-1}).$$

Expanding \square_g in this null frame (see [1]) gives

$$|\square_g \chi|^2 = |-\underline{L}L\chi + E_1^2\chi + E_2^2\chi + (\nabla_{\underline{L}}L - \nabla_{E_1}E_1 - \nabla_{E_2}E_2)\chi|^2 \lesssim 1_{\text{supp}(\nabla\chi)} r^{-2}.$$

In summary, we have

$$|(\square_g \chi) \psi_\epsilon|^2 + |\nabla^\mu \chi \nabla_\mu \psi_\epsilon|^2 \lesssim 1_{\text{supp}(\nabla\chi)} \left(\frac{|\psi_\epsilon|^2}{r^2} + |\partial \psi_\epsilon|^2 \right) \Rightarrow$$

$$\limsup_{\epsilon \rightarrow 0} \int_{\mathcal{B}} \sum_{m,l} \int_{r_+}^{\infty} |F_\epsilon|^2 r^2 dr d\omega \lesssim$$

$$\limsup_{\epsilon \rightarrow 0} \int_{(t,r,\theta,\phi)} 1_{\text{supp}(\nabla\chi)} \left(\frac{|\psi_\epsilon|^2}{r^2} + |\partial \psi_\epsilon|^2 \right) r^2 \sin \theta dt dr d\theta d\phi.$$

We conclude the proof by appealing to the Hardy inequality of appendix B to control the 0th order term, using finite in time (non-degenerate) energy estimates, and then taking ϵ to 0. \square

Now we can prove Theorem 1.7

Proof. By Plancherel, it suffices to prove

$$\int_{\mathcal{B}} \sum_{(m,l) \in \mathbb{C}} \left((|u(-\infty)|^2 + |u(\infty)|^2) + \int_{r_0}^{r_1} (|u'|^2 + |u|^2) dr^* \right) d\omega \leq$$

$$B(r_0, r_1, C_{\mathcal{B}}, C_{\mathbb{C}}) \int_{\Sigma_0} |\partial \psi|^2.$$

We begin with proposition 3.1 which gives

$$u_\epsilon(r^*) =$$

$$W_\epsilon^{-1} \left(u_{\text{out},\epsilon}(r^*) \int_{-\infty}^{r^*} u_{\text{hor},\epsilon}(x^*) H_\epsilon(x^*) dx^* + u_{\text{hor},\epsilon}(r^*) \int_{r^*}^{\infty} u_{\text{out},\epsilon}(x^*) H_\epsilon(x^*) dx^* \right).$$

This implies

$$|u(-\infty)|^2 \leq \limsup_{\epsilon \rightarrow 0} \left| W^{-1} \int_{-\infty}^{\infty} u_{\text{out},\epsilon}(x^*) H_{\epsilon}(x^*) dx^* \right|^2 \quad (3.2)$$

$$\int_{r_0/2}^{2r_1} |u|^2 dr^* \lesssim \limsup_{\epsilon \rightarrow 0} \left(|W|^{-2} \int_{r_+}^{r(A)} |F_{\epsilon}|^2 dr + \left| W^{-1} \int_A^{\infty} u_{\text{out},\epsilon}(x^*) H_{\epsilon}(x^*) dx^* \right|^2 \right) \quad (3.3)$$

where A is a sufficiently large constant, possibly depending on r_1 . Then, an elliptic estimates implies

$$\int_{r_0}^{r_1} |u'|^2 dr^* \lesssim \int_{r_0/2}^{2r_1} |u|^2 dr^* + \limsup_{\epsilon \rightarrow 0} \int_{r_+}^{\infty} |F_{\epsilon}|^2 dr. \quad (3.4)$$

Lastly, to control $|u(\infty)|^2$, we use the already introduced microlocal energy current:

$$\begin{aligned} \omega^2 |u(\infty)|^2 &= Q_T(\infty) = Q_T(-\infty) + \int_{-\infty}^{\infty} (Q_T)' dr^* \Rightarrow \\ |u(\infty)|^2 &\lesssim \omega(am - 2Mr_+\omega) |u(-\infty)|^2 + \omega \int_{-\infty}^{\infty} \text{Im}(H\bar{u}) dr^*. \end{aligned} \quad (3.5)$$

After applying Plancherel, the proof of Lemma 3.2, Theorem 1.4, and adding inequalities (3.2), (3.3), (3.4), and (3.5) together, we get

$$\begin{aligned} &\int_{\mathcal{B}} \sum_{(m,l) \in \mathcal{C}} \left(|u(-\infty)|^2 + |u(\infty)|^2 \right) d\omega + \int_{\mathcal{B}} \sum_{(m,l) \in \mathcal{C}} \int_{r_0}^{r_1} \left(|u'|^2 + |u|^2 \right) dr^* d\omega \\ &\lesssim \int_{\Sigma_0} |\partial\psi|^2 + \int_{\mathcal{B}} \sum_{(m,l) \in \mathcal{C}} \limsup_{\epsilon \rightarrow 0} \left| \int_A^{\infty} u_{\text{out},\epsilon}(x^*) H_{\epsilon}(x^*) dx^* \right|^2 d\omega. \end{aligned}$$

It just remains to control the last term on the right hand side. Note that a naive application of Cauchy-Schwarz would produce too many powers of x^* to finish the argument; however, if we somehow gained a power of x^{-1} we could always use the inequality

$$\limsup_{\epsilon \rightarrow 0} \left| \int_A^{\infty} u_{\text{out},\epsilon}(x^*) H_{\epsilon}(x^*) x^{-1} dx^* \right|^2 \lesssim \limsup_{\epsilon \rightarrow 0} \int_{r(A)}^{\infty} |F_{\epsilon}|^2 r^2 dr.$$

After integrating in ω and summing in (m,l) , this can be controlled by Lemma 3.2. We will denote by G all terms that can be controlled by this sort

of brute force Cauchy-Schwarz inequality. Let's return to the troublesome term. We start by observing that

$$\begin{aligned} & \int_{\mathbb{B}} \sum_{(m,l) \in \mathbb{C}} \limsup_{\epsilon \rightarrow 0} \left| \int_A^\infty u_{\text{out},\epsilon}(x^*) H_\epsilon(x^*) dx^* \right|^2 d\omega = \\ & \int_{\mathbb{B}} \sum_{(m,l) \in \mathbb{C}} \left(\limsup_{\epsilon \rightarrow 0} \left| \int_A^\infty e^{i\omega x^*} e^{-\epsilon x^*} H_\epsilon(x^*) dx^* \right|^2 + G \right) d\omega. \end{aligned}$$

The plan is to take advantage of the oscillations in ω by a suitable application of Plancherel. However, we will first need to account for all of the ω dependence in H_ϵ . Let's introduce the variables

$$\begin{aligned} u &:= \frac{1}{2}(t - r^*) \\ v &:= \frac{1}{2}(t + r^*). \end{aligned}$$

From the definitions of the cutoff and the triangle inequality, it follows that

$$\begin{aligned} & \left| \int_A^\infty e^{i\omega x^*} e^{-\epsilon x^*} H_\epsilon(x^*) dx^* \right|^2 \lesssim \\ & \left| \int_{\mathbb{S}^2} \int_{-\infty}^\infty \int_{\tilde{A}}^\infty e^{2i\omega v} e^{-2\epsilon v} (\partial_u \chi) (\partial_v \psi) e^{-im\phi} S_{ml}(\theta, \omega) r \sin \theta dv du d\theta d\phi \right|^2 + \\ & \left| \int_{\mathbb{S}^2} \int_{-\infty}^\infty \int_{\tilde{A}}^\infty e^{2i\omega v} e^{-2\epsilon v} (\square_g \chi) \psi e^{-im\phi} S_{ml}(\theta, \omega) r \sin \theta dv du d\theta d\phi \right|^2 + G \end{aligned}$$

Here \tilde{A} denotes a large fixed constant possibly depending on A . Let's focus on the first term on the right hand side since the second term will be treated similarly. Using Plancherel relative to the orthonormal basis $\{e^{im\phi} S_{ml}(\theta, a\omega)\}$ of $L^2(\sin \theta d\theta d\phi)$ gives

$$\begin{aligned} & \int_{\mathbb{B}} \sum_{(m,l) \in \mathbb{C}} \left| \int_{\mathbb{S}^2} \int_{-\infty}^\infty \int_{\tilde{A}}^\infty e^{2i\omega v} e^{-2\epsilon v} (\partial_u \chi) (\partial_v \psi) e^{-im\phi} S_{ml}(\theta, \omega) r \sin \theta dv du d\theta d\phi \right|^2 d\omega \\ & \lesssim \int_{\mathbb{S}^2} \int_{\mathbb{B}} \left| \int_{-\infty}^\infty \int_{\tilde{A}}^\infty e^{2i\omega v} e^{-2\epsilon v} (\partial_u \chi) (\partial_v \psi) r dv du \right|^2 d\omega \sin \theta d\theta d\phi. \quad (3.6) \end{aligned}$$

Due to the support of $\partial_u \chi$, the u integration occurs over a region of uniformly bounded size. Hence, Cauchy-Schwarz in the u integral implies that (3.6) is controlled by

$$\begin{aligned} \int_{\mathbb{S}^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| \int_{\tilde{A}}^{\infty} e^{2i\omega v} e^{-\epsilon(x^*+t)} \partial_u \chi \partial_v \psi r dv \right|^2 d\omega du \sin \theta d\theta d\phi \lesssim \\ \int_{\mathbb{S}^2} \int_{-\infty}^{\infty} \int_{\tilde{A}}^{\infty} |\partial_u \chi \partial_v \psi|^2 r^2 du dv \sin \theta d\theta d\phi \end{aligned}$$

Now we can just appeal to (the proof of) Lemma 3.2. For the term

$$\left| \int_{\mathbb{S}^2} \int_{-\infty}^{\infty} \int_{\tilde{A}}^{\infty} e^{2i\omega v} e^{-2\epsilon v} (\square_g \chi) \psi e^{-im\phi} S_{ml}(\theta, \omega) r \sin \theta dv du d\theta d\phi \right|^2$$

we can carry out exactly the same procedure except that we add a Hardy inequality (appendix B) at the end so that we can close the estimate at the level of derivatives of ψ . In conclusion, we have

$$\int_{\mathcal{B}} \sum_{(m,l) \in \mathcal{C}} \limsup_{\epsilon \rightarrow 0} \left| \int_A^{\infty} u_{\text{out},\epsilon}(x^*) H_{\epsilon}(x^*) dx^* \right|^2 d\omega \lesssim \int_{\Sigma_0} |\partial \psi|^2.$$

□

Before concluding the section, we would like to emphasize that for the applications to [10], it is crucial that we have arranged for the right hand side of this estimate be given by a non-degenerate energy flux through Σ_0 .

4 The Integral Transformation

In this section we will prove propositions 2.1, 2.2, and 2.3. For clarity of exposition we will restrict ourselves to $\omega \in \mathbb{R} \setminus \{0\}$; indeed, for $\text{Im}(\omega) > 0$ the proofs are much easier and follow from the same sort of reasoning as the real ω case. Furthermore, due to the symmetries of the radial ODE, we may restrict ourselves to $\omega > 0$.

Proof. To verify \tilde{u} 's equation it is useful to consider the following functions

$$\begin{aligned} g(r) &:= (r - r_+)^{-\xi} (r - r_-)^{-\eta} e^{i\omega r} R(r), \\ \tilde{g}(z) &:= \int_{r_+}^{\infty} e^{\frac{2i\omega}{r_+ - r_-} (z - r_-)(r - r_-)} (r - r_-)^{2\eta} (r - r_+)^{2\xi} e^{-2i\omega r} g(r) dr \end{aligned}$$

$$= \int_{r_+}^{\infty} e^{\frac{2i\omega}{r_+ - r_-}(z - r_-)(r - r_-)} (r - r_-)^{\eta} (r - r_+)^{\xi} e^{-i\omega r} R(r) dr.$$

Here $z = x + iy$ with $y \geq 0$. Since $\omega > 0$, the integrals are absolutely convergent if $y > 0$. We say that a function h satisfies a Confluent Heun Equation (CHE) if there are complex parameters $\gamma, \delta, p, \alpha$, and σ such that

$$\begin{aligned} Th := (r - r_+)(r - r_-) \frac{d^2 h}{dr^2} + (\gamma(r - r_+) + \delta(r - r_-) + p(r - r_+)(r - r_-)) \frac{dh}{dr} + \\ (\alpha p(r - r_-) + \sigma) h = G. \end{aligned} \quad (4.1)$$

One finds that g satisfies such a CHE with

$$\begin{aligned} \gamma &= 2\eta + 1, \\ \delta &= 2\xi + 1, \\ p &= -2i\omega, \\ \alpha &= 1, \\ \sigma &= 2am\omega - 2\omega r_- i - \lambda_{ml} - a^2 \omega^2, \\ G &= (r - r_+)^{-\xi} (r - r_-)^{-\eta} e^{i\omega r} \hat{F}. \end{aligned}$$

Now we need an integration by parts lemma whose straightforward proof is omitted.

Lemma 4.1. *Let T denote a Confluent Heun operator as defined in (4.1). Then*

$$\begin{aligned} \int_{\alpha}^{\beta} (Tf) (r - r_+)^{\delta-1} (r - r_-)^{\gamma-1} e^{pr} h dr = \\ (r - r_+)^{\delta} (r - r_-)^{\gamma} e^{pr} \left(\frac{df}{dr} h - f \frac{dh}{dr} \right) \Big|_{\alpha}^{\beta} + \int_{\alpha}^{\beta} (Th) (r - r_+)^{\delta-1} (r - r_-)^{\gamma-1} e^{pr} f dr. \end{aligned}$$

Since the coefficients of the CHE are all holomorphic, we may take the derivatives in the CHE to be complex derivatives. Let L_r denote a Confluent Heun Operator in the r variable with parameters given above. Let \tilde{L}_z denote a Confluent Heun operator in the $z (= x + iy)$ variable with, to be determined, tilded parameters. We wish to determine $A \in \mathbb{C}$ such that

$$\int_{r_+}^{\infty} e^{A(z - r_-)(r - r_-)} (r - r_-)^{2\eta} (r - r_+)^{2\xi} e^{-2i\omega r} g(r) dr$$

is a solution to a CHE with tilded parameters for $y > 0$. If the integral is appropriately convergent to allow differentiation under the integral sign, we see from Lemma 4.1 that the following two conditions will suffice:

$$\left(\tilde{L}_z - L_r\right) e^{A(z-r_-)(r-r_-)} = 0,$$

$$(r-r_+)^{\delta}(r-r_-)^{\gamma} e^{pr} e^{A(z-r_-)(r-r_-)} \left(A(z-r_-)g - \frac{dg}{dr}\right) \Big|_{r_+}^{\infty} = 0 \quad \forall z \text{ such that } y > 0.$$

We have

$$\begin{aligned} & e^{-A(z-r_-)(r-r_-)} \left(\tilde{L}_z - L_r\right) e^{A(z-r_-)(r-r_-)} = \\ & A(A(r_+ - r_-) + \tilde{p})(r-r_-)(z-r_-)^2 - A(A(r_+ - r_-) + p)(r-r_-)^2(z-r_-) + \\ & -A\left(\gamma + \delta + p(r_+ - r_-) - \tilde{\gamma} - \tilde{\delta} - \tilde{p}(r_+ - r_-)\right)(z-r_-)(r-r_-) + \\ & (A\gamma(r_+ - r_-) + \tilde{\alpha}\tilde{p})(z-r_-) - (A\tilde{\gamma}(r_+ - r_-) + \alpha p)(r-r_-) + (\tilde{\sigma} - \sigma). \end{aligned}$$

From this it is clear that we must have

$$A = -p(r_+ - r_-)^{-1} = 2i\omega(r_+ - r_-)^{-1},$$

$$\tilde{p} = p = -2i\omega,$$

$$\tilde{\alpha} = \gamma,$$

$$\tilde{\gamma} = \alpha = 1,$$

$$\tilde{\delta} = \gamma + \delta - \tilde{\gamma} = 1 - 4iM\omega,$$

$$\tilde{\sigma} = \sigma.$$

We still need to check that the boundary conditions are satisfied. Since g and $\frac{dg}{dr}$ both decay for large r , the exponential decay from $e^{A(z-r_-)(r-r_-)}$ clearly implies that

$$\left((r-r_+)^{\delta}(r-r_-)^{\gamma} e^{pr} e^{A(z-r_-)(r-r_-)} \left(A(z-r_-)g - \frac{dg}{dr}\right)\right)(r=\infty) = 0$$

for all z with $y > 0$.

Since $\delta = 2\xi + 1$, with ξ purely imaginary, and $|g|$ extends continuously to r_+ , we see that

$$\left((r-r_+)^{\delta}(r-r_-)^{\gamma} e^{pr} e^{A(z-r_-)(r-r_-)} \left(A(z-r_-)g - \frac{dg}{dr}\right)\right)(r=r_+) = 0 \Leftrightarrow$$

$$\frac{dg}{dr^*}(r_+) = 0.$$

If we r^* differentiate the expression defining g , we get

$$\left| \frac{dg}{dr^*} \right| (r_+) = \left| \frac{dR}{dr^*} - \frac{\xi(r_+ - r_-)}{2Mr_+} R \right| (r_+) = 0.$$

Recall that for a holomorphic function, $\frac{d}{dz} = \frac{\partial}{\partial x}$. We can thus conclude

Lemma 4.2. *If $y > 0$ we have*

$$\begin{aligned} & (z - r_+)(z - r_-) \frac{\partial^2 \tilde{g}}{\partial x^2} + \\ & ((z - r_+) + (1 - 4iM\omega)(z - r_-) - 2i\omega(z - r_-)(z - r_+)) \frac{\partial \tilde{g}}{\partial x} + \\ & (-2i\omega(2\eta + 1)(z - r_-) + 2am\omega - 2\omega r_- i - \lambda_{ml} - a^2\omega^2) \tilde{g} = \tilde{G} \end{aligned}$$

where

$$\tilde{G} := \int_{r_+}^{\infty} e^{\frac{2i\omega}{r_+ - r_-}(z - r_-)(r - r_-)} (r - r_-)^{2\eta} (r - r_+)^{2\xi} e^{-2i\omega r} G(r) dr.$$

Next, we would like to take y to 0. However, the integrals defining $\frac{\partial \tilde{g}}{\partial x}$ are not a priori convergent when $y = 0$. The following lemma will give us the necessary control.

Lemma 4.3. *\tilde{g} is in $H_{[r_+, \infty)}^1(\cdot + iy)$, uniformly in y . More specifically, we have*

1. \tilde{g} is uniformly bounded.
2. $\tilde{g} = x^{-1} ((\cdot - r_+)^{-\xi} R(\cdot)) (r_+) \varphi(z) + O\left(\frac{\log(x)}{x^{-2}}\right)$ for large x .
3. $\frac{\partial \tilde{g}}{\partial x}$ is uniformly bounded.
4. $\frac{\partial \tilde{g}}{\partial x} - 2i\omega \tilde{g} = O(x^{-2})$ for large x .

All constants are uniform in y , and $\varphi(z)$ is a function with

$$|\varphi(z)| = e^{-2\omega y \frac{r_+ - r_-}{2\omega}}.$$

Proof. Let $\epsilon > 0$ be arbitrary. Then integrating by parts twice gives

$$\begin{aligned}\tilde{g}(z) &= \int_{r_+}^{r_++\epsilon} e^{A(z-r_-)(r-r_-)} (r-r_-)^\eta (r-r_+)^\xi e^{-i\omega r} R(r) dr + \\ &- (A(z-r_-))^{-1} e^{A(z-r_-)(r_+-r_-+\epsilon)} (r_+-r_-+\epsilon)^\eta \epsilon^{2\xi} e^{-i\omega(r_++\epsilon)} \left((\cdot - r_+)^{-\xi} R(\cdot) \right) (r_++\epsilon) + \\ &(A(z-r_-))^{-2} e^{A(z-r_-)(r_+-r_-+\epsilon)} \frac{d}{dr} \left((r-r_-)^\eta (r-r_+)^\xi e^{-i\omega r} R(r) \right) (r_++\epsilon) + \\ &(A(z-r_-))^{-2} \int_{r_++\epsilon}^{\infty} e^{A(z-r_-)(r-r_-)} \frac{d^2}{dr^2} \left((r-r_-)^\eta (r-r_+)^\xi e^{-i\omega r} R(r) \right) dr.\end{aligned}$$

Near r_+ , the boundary conditions for R give

$$\left| \frac{d^k}{dr^k} \left((r-r_-)^\eta (r-r_+)^\xi e^{-i\omega r} R(r) \right) \right| \lesssim (r-r_+)^{-k}.$$

Away from r_+ , the boundary conditions for R give

$$\left| \frac{d^k}{dr^k} \left((r-r_-)^\eta (r-r_+)^\xi e^{-i\omega r} R(r) \right) \right| \lesssim r^{-k-1}.$$

Let's set $\epsilon = x^{-2}$ and examine each term in the sum above. We have

$$\int_{r_+}^{r_++x^{-2}} e^{A(z-r_-)(r-r_-)} (r-r_-)^\eta (r-r_+)^\xi e^{-i\omega r} R(r) dr = O(x^{-2}).$$

Next, we have

$$\begin{aligned}&- (A(z-r_-))^{-1} e^{A(z-r_-)(r_+-r_-+x^{-2})} (r_+-r_-+x^{-2})^\eta x^{-4\xi} e^{-i\omega(r_++x^{-2})} \times \\ &\quad \left((\cdot - r_+)^{-\xi} R(\cdot) \right) (r_++x^{-2}) = \\ &-A^{-1} e^{A(z-r_-)(r_+-r_-)} (r_+-r_-)^\eta e^{-i\omega r_+} \left((\cdot - r_+)^{-\xi} R(\cdot) \right) (r_+) x^{-(4\xi+1)} + O(x^{-2}).\end{aligned}$$

The third term does not decay as fast

$$\begin{aligned}&(A(z-r_-))^{-2} e^{A(z-r_-)(r_+-r_-+x^{-2})} \frac{d}{dr} \left((r-r_-)^\eta (r-r_+)^\xi e^{-i\omega r} R(r) \right) (r_++x^{-2}) = \\ &2\xi A^{-2} e^{A(z-r_-)(r_+-r_-+x^{-2})} (r_+-r_-)^\eta e^{-i\omega r_+} \left((\cdot - r_+)^{-\xi} R(\cdot) \right) (r_+) x^{-4\xi} + O(x^{-2}).\end{aligned}$$

However, this will cancel with the fourth term

$$\begin{aligned}
& (A(z - r_-))^{-2} \int_{r_+ + x^{-2}}^{\infty} e^{A(z-r_-)(r-r_-)} \frac{d^2}{dr^2} \left((r - r_-)^\eta (r - r_+)^\xi e^{-i\omega r} R(r) \right) dr = \\
& (A(z - r_-))^{-2} \int_{r_+ + x^{-2}}^1 e^{A(z-r_-)(r-r_-)} \frac{d^2}{dr^2} \left((r - r_-)^\eta (r - r_+)^\xi e^{-i\omega r} R(r) \right) dr + O(x^{-2}) = \\
& \quad (A(z - r_-))^{-2} e^{A(z-r_-)(r_+ - r_-)} (r_+ - r_-)^\eta \times \\
& \quad \left(\left((\cdot - r_+)^{-\xi} R(\cdot) \right) (r_+) (2\xi)(2\xi - 1) \int_{r_+ + x^{-2}}^1 (r - r_+)^{2\xi - 2} dr \right) + O\left(\frac{\log(x)}{x^{-2}}\right) = \\
& -2\xi A^{-2} e^{A(z-r_-)(r_+ - r_- + x^{-2})} (r_+ - r_-)^\eta e^{-i\omega r_+} \left((\cdot - r_+)^{-\xi} R(\cdot) \right) (r_+) x^{-4\xi} + O\left(\frac{\log(x)}{x^{-2}}\right).
\end{aligned}$$

Adding all the terms together implies that

$$|\tilde{g}(z)| = e^{-2\omega y} \frac{r_+ - r_-}{2\omega} |R(r_+)| x^{-1} + O\left(\frac{\log(x)}{x^{-2}}\right).$$

Thus we have established assertions 1 and 2. Next, we see that

$$\frac{\partial \tilde{g}}{\partial x} - A(r_+ - r_-) \tilde{g} = A \int_{r_+}^{\infty} e^{A(z-r_-)(r-r_-)} (r - r_-)^\eta (r - r_+)^{\xi+1} e^{-i\omega r} R(r) dr.$$

The point here is that the first integration by parts will produce no boundary term at r_+ . Letting $\epsilon > 0$ and integrating by parts three times then gives

$$\begin{aligned}
& \frac{\partial \tilde{g}}{\partial x} - A(r_+ - r_-) \tilde{g} = \\
& -(z - r_-)^{-1} \int_{r_+}^{r_+ + \epsilon} e^{A(z-r_-)(r-r_-)} \frac{d}{dr} \left((r - r_-)^\eta (r - r_+)^{\xi+1} e^{-i\omega r} R(r) \right) dr + \\
& A^{-1} (z - r_-)^{-2} e^{A(z-r_-)(r_+ - r_- + \epsilon)} \frac{d}{dr} \left((r - r_-)^\eta (r - r_+)^{\xi+1} e^{-i\omega r} R(r) \right) (r_+ + \epsilon) + \\
& -A^{-2} (z - r_-)^{-3} e^{A(z-r_-)(r_+ - r_- + \epsilon)} \frac{d^2}{dr^2} \left((r - r_-)^\eta (r - r_+)^{\xi+1} e^{-i\omega r} R(r) \right) (r_+ + \epsilon) + \\
& -A^{-2} (z - r_-)^{-3} \int_{r_+ + \epsilon}^{\infty} e^{A(z-r_-)(r-r_-)} \frac{d^3}{dr^3} \left((r - r_-)^\eta (r - r_+)^{\xi+1} e^{-i\omega r} R(r) \right) dr.
\end{aligned}$$

Setting $\epsilon = x^{-1}$ establishes assertions 3 and 4. \square

With this lemma it is now clear that we may take y to 0 and get

$$\begin{aligned} & (x - r_+)(x - r_-) \frac{d^2 \tilde{g}}{dx^2} + \\ & ((x - r_+) + (1 - 4iM\omega)(x - r_-) - 2i\omega(x - r_-)(x - r_+)) \frac{d\tilde{g}}{dx} + \\ & (-2i\omega(2\eta + 1)(x - r_-) + 2am\omega - 2\omega r_- i - \lambda_{ml} - a^2\omega^2) \tilde{g} = \tilde{G}. \end{aligned}$$

From here it is simply a matter of some tedious algebra to establish proposition 2.1. It is similarly straightforward to establish proposition 2.3 from Lemma 4.3. \square

5 Some Estimates for the Kerr ODE

For the purposes of section 2 we need to prove proposition 2.5:

$$|u(-\infty)|^2 \lesssim (4\epsilon)^{-1} \int_{r_+}^{\infty} |F(r)|^2 r^4 dr + \epsilon \int_{r_+}^{\infty} |R(r)|^2 dr \quad \forall \epsilon > 0 \Rightarrow \quad (5.1)$$

$$|u(-\infty)|^2 \lesssim \int_{r_+}^{\infty} |F(r)|^2 r^4 dr. \quad (5.2)$$

It will sometimes be useful to switch our perspectives on $-\infty$ and ∞ and write

$$u'' + (\omega_0^2 - V_0) u = H$$

where

$$\omega_0 = \omega - \frac{am}{2Mr_+},$$

$$V_0 = V + \omega_0^2 - \omega^2.$$

For the following estimates the relevant properties of V and V_0 are

1. V is uniformly bounded.
2. $V = O(r^{-2})$ at ∞ .
3. $V_0 = O(r - r_+)$.
4. For fixed non-zero a and m , there exists a constant $c > 0$ such that $am - 2Mr_+\omega \geq -c(\lambda_{ml} + a^2\omega^2) \Rightarrow \frac{dV_0}{dr}(r_+) > 0$.

The last statement is the only non-obvious one, and the relevant computations can be found in [10]. It will also be useful to note that

$$\lambda_{ml} + a^2\omega^2 \geq |m|(|m| + 1).$$

This follows from the observation that when $a^2\omega^2 = 0$, the $e^{im\phi}S_{ml}(\theta)$ are simply spherical harmonics with corresponding eigenvalues all larger than $|m|(|m| + 1)$.

We will explore various estimates and their realm of applicability. Then at the end we will show how they can be combined to establish (5.2). We will borrow the “separated current template” from [10].

5.1 Virial Estimate I

The estimates of this section require that ω be bounded away from 0 and that we have a priori control of $Q_T(\infty)$. The resulting estimate will be sufficiently good near ∞ , but will require strengthening near $-\infty$.

The virial current is

$$Q^y := y|u'|^2 + y(\omega^2 - V)|u|^2$$

where y is a suitably chosen function. We have

$$(Q^y)' = y'|u'|^2 + y'\omega^2|u|^2 - (yV)'|u|^2 + 2y\operatorname{Re}(H\overline{u}').$$

Integrating this gives

$$\begin{aligned} & \int_{-\infty}^{\infty} (y'|u'|^2 + y'\omega^2|u|^2 - (yV)'|u|^2) dr^* = \\ & Q^y(\infty) - Q^y(-\infty) - \int_{-\infty}^{\infty} 2y\operatorname{Re}(u'\overline{H}) dx^*. \end{aligned}$$

We want to choose y so that the left hand side controls $|u|^2 + |u'|^2$ (possibly with weights), and so that the boundary terms are controllable. Let $\zeta(r^*)$ be a non-negative function which is identically 1 near $r^* = -\infty$ and equals r^{-2} near $r^* = \infty$. We set

$$y(r^*) := \exp\left(-B \int_{r^*}^{\infty} \zeta dr^*\right).$$

Here B is a large parameter to be chosen later. We have $y(r_+) = 0$, $y(\infty) = 1$, and $y' = B\zeta y > 0$. We will show that the term

$$- \int_{-\infty}^{\infty} (yV)'|u|^2 dr^*$$

which threatens to destroy the coercivity of our estimate can in fact be absorbed into the other two terms. After an integration by parts and the inequality $|ab| \leq \epsilon|a| + (4\epsilon)^{-1}|b|$, we find

$$\left| \int_{-\infty}^{\infty} (yV)' |u|^2 dr^* \right| \leq \frac{1}{2} \int_{-\infty}^{\infty} y' |u'|^2 dr^* + 2 \int_{-\infty}^{\infty} (y'\omega^2) \frac{y^2 |V|^2}{\omega^2 (y')^2} |u|^2 dr^* + \left| (yV|u|^2) \Big|_{-\infty}^{\infty} \right|.$$

Note that $|V|$ is uniformly bounded, decays like r^{-2} , and that $y/y' \leq B^{-1}r^2$. Also, the boundary terms clearly vanish. Thus, for sufficiently large B , we get

$$\left| \int_{-\infty}^{\infty} (yV)' dr^* \right| \leq \frac{1}{2} \int_{-\infty}^{\infty} (y' |u'|^2 + y' \omega^2 |u|^2) dr^*.$$

Lastly, we note that

$$Q^y(\infty) = 2Q_T(\infty).$$

Thus, we end up with

$$\int_{-\infty}^{\infty} (y' |u'|^2 + y' \omega^2 |u|^2) dr^* \lesssim Q_T(\infty) - \int_{-\infty}^{\infty} y \operatorname{Re}(u' \overline{H}) dr^*. \quad (5.3)$$

The usual Cauchy-Schwarz argument then gives

$$\int_{-\infty}^{\infty} (y' |u'|^2 + y' \omega^2 |u|^2) dr^* \lesssim |Q_T(\infty)| + \int_{-\infty}^{\infty} y |H|^2 r^2 dr^*. \quad (5.4)$$

This estimate is sufficiently strong away from the horizon. However, near $-\infty$, the exponential decay of the weight y makes the estimate quite weak.

5.2 Virial Estimate II

In this section we look at the virial current from the opposite direction. This estimate will require that ω_0 is bounded away from 0 and that we have a priori control of $Q_T(-\infty)$. The resulting estimate will be sufficiently strong near r_+ , but will require strengthening near ∞ .

We rewrite the virial current as

$$Q^y := y |u'|^2 + y (\omega_0^2 - V_0) |u|^2.$$

Let $\zeta(r)$ be a positive function equal to Δ near $r = r_+$, and equal to 1 near $r = \infty$. Then define

$$y(r^*) := \exp \left(-B \int_{-\infty}^{r^*} \zeta dr^* \right).$$

Integrating the virial current gives

$$\begin{aligned} \int_{-\infty}^{\infty} (-y'|u'|^2 - y'\omega_0^2|u|^2 + (yV_0)'|u|^2) dr^* = \\ -Q^y(\infty) + Q^y(-\infty) + \int_{-\infty}^{\infty} 2y\operatorname{Re}(u'\overline{H}) dx^*. \end{aligned}$$

We may deal with the $(yV_0)'$ exactly as in the previous section. This time

$$Q^y(\infty) = 0$$

$$Q^y(-\infty) \approx 2\frac{\omega_0}{\omega}Q_T(-\infty).$$

We end up with

$$\int_{-\infty}^{\infty} (-y'|u'|^2 - y'\omega_0^2|u|^2) dr^* \lesssim -\frac{\omega_0}{\omega}Q_T(-\infty) + \int_{-\infty}^{\infty} y\operatorname{Re}(u'\overline{H}) dr^*. \quad (5.5)$$

As in the previous section, it is clear that we also have

$$\int_{-\infty}^{\infty} (-y'|u'|^2 - y'\omega_0^2|u|^2) dr^* \lesssim \left| \frac{\omega_0}{\omega}Q_T(-\infty) \right| + \int_{r_+}^{\infty} |F|^2 dr. \quad (5.6)$$

This estimate is sufficiently strong away from ∞ . However, near ∞ , the exponential decay of the weight y makes the estimate very weak.

5.3 The Red Shift Estimate

The estimate of this section will require that $\frac{dV_0}{dr}(r_+) > 0$ and that we can already estimate

$$\int_{\alpha}^{\beta} (|u'|^2 + |u|^2) dr^*$$

for arbitrary $r_+ < \alpha < \beta < \infty$.

The following Poincaré type inequality will be useful.

Lemma 5.1. *Suppose h has support in $[r_+, r_+ + \epsilon]$ and has $\left((\cdot - r_+) |h|^2(\cdot)\right)(r_+) = 0$. Then*

$$\int_{r_+}^{\infty} |h|^2 dr \leq C(\epsilon) \int_{r_+}^{\infty} |h' + i\omega_0 h|^2 dr$$

where

$$C(\epsilon) \lesssim (1 + \epsilon^2).$$

Proof. We have

$$\begin{aligned} \int_{r_+}^{\infty} |h|^2 dr &= \int_{r_+}^{\infty} \frac{d}{dr} (r - r_+) |h|^2 dr = - \int_{r_+}^{\infty} (r - r_+) \left(\frac{dh}{dr} \bar{h} + h \frac{d\bar{h}}{dr} \right) dr = \\ &= - \int_{r_+}^{\infty} \left(\frac{r^2 + a^2}{r - r_-} \right) \left(h' \bar{h} + h \bar{h}' \right) dr = \\ &= - \int_{r_+}^{\infty} \left(\frac{r^2 + a^2}{r - r_-} \right) \left((h' + i\omega_0 h) \bar{h} + h (\bar{h}' - i\omega_0 \bar{h}) \right) dr. \end{aligned}$$

From here the lemma follows by the usual argument. \square

The (microlocal) red shift current is

$$Q_{\text{red}}^z := z |u' + i\omega_0 u|^2 - z V_0 |u|^2 = Q^z + 2z \frac{\omega_0}{\omega} Q_T.$$

Note that the boundary conditions for R imply that $(u' + i\omega_0 u)(r^*) = O(r - r_+)$ near $r^* = -\infty$. Hence, we may take z to be a function which blows up at $-\infty$. We have

$$(Q_{\text{red}}^z)' = z' |u' + i\omega_0 u|^2 - (z V_0)' |u|^2 + 2z \text{Re} \left((u' + i\omega_0 u) \bar{H} \right).$$

Let $\zeta(r)$ be a bump function identically 1 on $[r_+, r_+ + \epsilon]$ and vanishing on $[r_+ + 2\epsilon, \infty)$. ϵ is a free parameter that we will later take sufficiently small. Now set

$$z(r^*) := - \frac{\zeta(r(r^*))}{V_0}.$$

Note that $z' > 0$ near $-\infty$ since $\frac{d}{dr} V_0(-\infty) > 0$. We have

$$(Q_{\text{red}}^z) \Big|_{-\infty}^{\infty} = -|u(-\infty)|^2$$

which has a good sign. For $r \in [r_+, r_+ + \epsilon]$, we have

$$(Q_{\text{red}}^z)' = z' |u' + i\omega_0 u|^2 + 2z \text{Re} \left((u' + i\omega_0 u) \bar{H} \right).$$

Note that we have $z' \sim (r - r_+)^{-1}$ in this region. For $r \in [r_+ + \epsilon, r_+ + 2\epsilon]$ we will treat everything as an error:

$$|(Q_{\text{red}}^z)| \lesssim (|u'|^2 + |u|^2) + |z \text{Re} \left((u' + i\omega_0 u) \bar{H} \right)|.$$

Of course for $r \geq r_+ + 2\epsilon$ we have $(Q_{\text{red}}^z)' = 0$. Putting everything together will produce an estimate for

$$\int_{r_+}^{r_+ + \epsilon} (r - r_+)^{-2} |u'(r^*(r)) + i\omega_0 u(r^*(r))|^2 dr.$$

For ϵ sufficiently small, an application of Lemma 5.1 will show that this controls

$$\int_{r_+}^{r_++\epsilon/2} |u(r^*(r))|^2 dr$$

at the expense of introducing error terms

$$\int_{r_++\epsilon/2}^{r_++\epsilon} (|u'(r^*(r))|^2 + |u(r^*(r))|^2) dr.$$

We end up with

$$\begin{aligned} & \int_{r_+}^{r_++\epsilon} (r - r_+)^{-2} |u'(r^*(r)) + i\omega_0 u(r^*(r))|^2 dr + \int_{r_+}^{r_++\epsilon/2} |u(r^*(r))|^2 dr \lesssim \\ & \int_{\epsilon/2}^{2\epsilon} (|u|^2 + |u'|^2) dr^* + \int_{-\infty}^{\infty} |z \operatorname{Re}((u' + i\omega_0 u) \overline{H})| dr^*. \end{aligned} \quad (5.7)$$

As usual, this implies

$$\begin{aligned} & \int_{r_+}^{r_++\epsilon} (r - r_+)^{-2} |u'(r^*(r)) + i\omega_0 u(r^*(r))|^2 dr + \int_{r_+}^{r_++\epsilon/2} |u(r^*(r))|^2 dr \lesssim \\ & \int_{\epsilon/2}^{2\epsilon} (|u(r^*(r))|^2 + |u'(r^*(r))|^2) dr^* + \int_{-\infty}^{\infty} |F(r)|^2 dr. \end{aligned} \quad (5.8)$$

Note that for every fixed m and l , ϵ can be assumed to depend continuously on a and ω . This estimate is good near $-\infty$, but clearly is not sufficient otherwise.

5.4 Proof of Proposition 2.5

Let b_0 be a sufficiently small and $b_1 = 2b_0$. First we apply virial estimate I and conclude

$$\int_{b_0}^{\infty} \left(|R|^2 + \left| \frac{dR}{dr} \right|^2 \right) dr \lesssim |Q_T(\infty)| + \int_{r_+}^{\infty} |F|^2 r^4 dr. \quad (5.9)$$

Now, depending on whether ω_0 is small or large, we either carry out virial estimate II or the red shift estimate and combine with 5.9 to get

$$\int_{r_+}^{\infty} |R|^2 dr \lesssim |Q_T(-\infty)| + |Q_T(\infty)| + \int_{r_+}^{\infty} |F|^2 dr.$$

Next, we recall that the energy current $Q_T = \omega \text{Im}(u' \bar{u})$ satisfies

$$\begin{aligned} (Q_T)' &= \omega \text{Im}(H \bar{u}) \Rightarrow \\ |Q_T(\infty)| &\leq |Q_T(-\infty)| + \int_{r_+}^{\infty} (r^2 + a^2) |F| |R| dr \lesssim \\ &\epsilon^{-1} \int_{r_+}^{\infty} |F(r)|^2 r^4 dr + \epsilon \int_{-\infty}^{\infty} |R(r)|^2 dr \end{aligned}$$

where we have used (5.1) in the last line. Taking ϵ small enough, we may combine the various estimates to conclude

$$\int_{r_+}^{\infty} |R|^2 dr \lesssim \int_{r_+}^{\infty} |F(r)|^2 r^4 dr.$$

Reapplying the energy estimate finally implies

$$|u(-\infty)|^2 \approx |Q_T(-\infty)| \lesssim \int_{r_+}^{\infty} |F(r)|^2 r^4 dr.$$

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A Asymptotic Analysis of the Radial ODE

We will collect various facts concerning the radial ODE:

$$u'' + (\omega^2 - V) u = 0 \text{ for } \omega \in \mathbb{C} \setminus \{0\}.$$

The material in this section is standard, and the necessary background can be found in most textbooks on the asymptotic analysis of ODEs, e.g. [24].

When recast in the r variable our ODE has a regular singularity at r_+ . Finding the roots of the indicial equation allows us to uniquely define two linearly independent functions u_{hor} (already given by definition 1.2) and $u_{\text{hor}2}$ by

Definition A.1. *Let $u_{\text{hor}2}(r^*)$ be the unique function satisfying*

$$1. \quad u_{\text{hor}2}'' + (\omega^2 - V) u_{\text{hor}2} = 0.$$

$$2. \ u_{hor2} \sim (r - r_+)^{\frac{-i(am-2Mr_+\omega)}{r_+-r_-}} \text{ near } r^* = -\infty.$$

$$3. \ \left| (r - r_+)^{\frac{i(am-2Mr_+\omega)}{r_+-r_-}} u_{hor2}(-\infty) \right|^2 = 1.$$

Since we have a regular singularity, the “ \sim ” means that

$$u_{hor}(r^*)(r - r_+)^{\frac{-i(am-2Mr_+\omega)}{r_+-r_-}}$$

is holomorphic in r near r_+ . In fact, it can be given by an explicit power series which exhibits holomorphic dependence on ω . Analogous statements hold for u_{hor2} .

Our ODE has an irregular singularity at ∞ . Nevertheless, we can uniquely define two linearly independent functions u_{in} and u_{out} (already given by definition 1.3) by

Definition A.2. Let $u_{in}(r^*)$ be the unique function satisfying

$$1. \ u_{in}'' + (\omega^2 - V) u_{out} = 0.$$

$$2. \ u_{in} \sim e^{-i\omega r^*} \text{ near } r^* = \infty.$$

$$3. \ |(e^{i\omega r^*} u_{in})(\infty)|^2 = 1.$$

Since our singularity is irregular, “ \sim ” must be interpreted as follows: There exists explicit constants $\{C_i^{(in)}\}_{i=1}^\infty$ and $\{C_i^{(out)}\}_{i=1}^\infty$ such that for every $N \geq 1$

$$u_{in}(r^*) = e^{-i\omega r^*} \left(1 + \sum_{i=1}^N \frac{C_i^{in}}{(r^*)^i} \right) + O\left((r^*)^{-N-1}\right) \text{ for large } r^*,$$

$$u_{out}(r^*) = e^{i\omega r^*} \left(1 + \sum_{i=1}^N \frac{C_i^{out}}{(r^*)^i} \right) + O\left((r^*)^{-N-1}\right) \text{ for large } r^*.$$

It is important to note that the constants in these O ’s can be estimated explicitly if desired. By examining the construction of u_{out} , one finds that u_{out} will be holomorphic in ω in the upper half plane and smooth in ω in $\mathbb{R} \setminus \{0\}$. See [18] for a detailed discussion of the holomorphic dependence on ω .

B A Hardy Inequality

The following standard Hardy inequality will be useful.

Lemma B.1. *Suppose that $h(v)$ is in $C^1([0, \infty))$. Then,*

$$\left| \left(\cdot |h(\cdot)|^2 \right) (\infty) \right| = 0 \Rightarrow$$

$$\int_0^\infty |h(v)|^2 dv \lesssim \int_0^\infty |\partial_v h(v)|^2 v^2 dv.$$

Proof. We may define a $C^1((-\infty, \infty))$ extension of h by setting

$$\tilde{h}(v) = -3h(-v) + 4h\left(\frac{-v}{2}\right) \text{ for } v < 0,$$

$$\tilde{h}(v) = h(v) \text{ for } v \geq 0.$$

We extend in this fashion so that

$$\int_{-\infty}^\infty |\tilde{h}|^2 dv \approx \int_{r_+}^\infty |h|^2 dv,$$

$$\int_{-\infty}^\infty |\partial_v \tilde{h}|^2 v^2 dv \approx \int_{r_+}^\infty |\partial_v h|^2 v^2 dv.$$

Thus, it suffices to prove the proposition with \tilde{h} instead of h . The standard argument goes

$$\int_{-\infty}^\infty |\tilde{h}|^2 dv = \int_{-\infty}^\infty \partial_v(v) |\tilde{h}|^2 dv \lesssim \int_{-\infty}^\infty |\tilde{h} \partial_v \tilde{h} v| dv \lesssim$$

$$\|\tilde{h}\|_{L^2} \left\| \partial_v \tilde{h} v \right\|_{L^2}$$

Dividing through by $\|\tilde{h}\|_{L^2}$ and squaring finishes the proof. \square

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